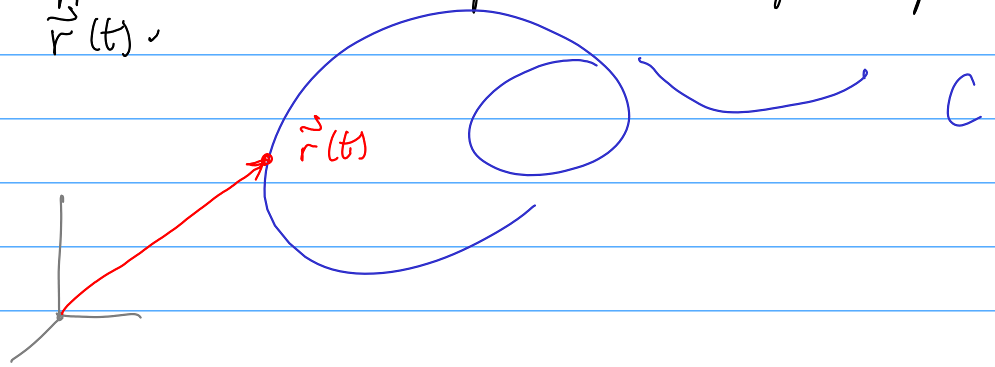
§13.2 ~ Calculus w/ Space curves

1/28/20

Suppose we have a space curve C given by a vector function $\vec{r}(t)$.



<u>Defn.</u> The <u>derivative</u> of $\vec{r}$ is the vector defined by

$$\vec{r}\,'(t) = \dot{\vec{r}}(t) = \frac{d\vec{r}}{dt} = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \lim_{t\to a} \frac{\vec{r}(t) - \vec{r}(a)}{t-a}$$

provided the limit exists.

<u>Thm.</u> If $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, then $\dot{\vec{r}}(t) = \langle \dot{x}(t), \dot{y}(t), \dot{z}(t)\rangle$.

<u>Proof.</u> $\dot{\vec{r}}(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \lim_{h\to 0} \frac{\langle x(t+h), y(t+h), z(t+h)\rangle - \langle x(t), y(t), z(t)\rangle}{h}$

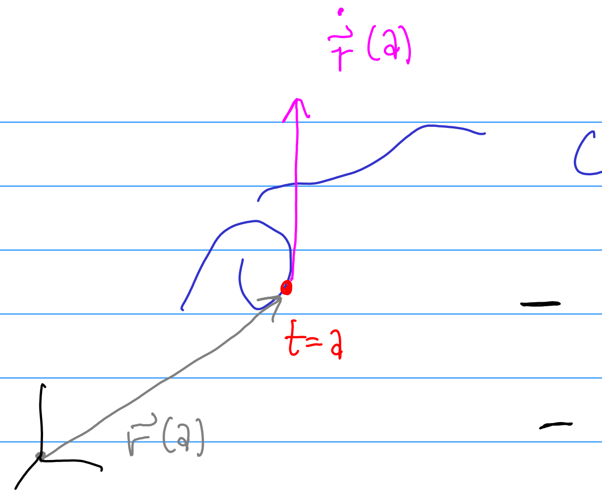
$$= \lim_{h\to 0} \frac{1}{h}\langle x(t+h) - x(t), y(t+h) - y(t), z(t+h) - z(t)\rangle$$

$$= \left\langle \underbrace{\lim_{h\to 0} \frac{x(t+h) - x(t)}{h}}_{\dot{x}(t)}, \lim_{h\to 0} \frac{y(t+h) - y(t)}{h}, \lim_{h\to 0} \frac{z(t+h) - z(t)}{h} \right\rangle$$

$$= \langle \dot{x}(t), \dot{y}(t), \dot{z}(t)\rangle \quad \blacksquare$$

<u>Corollary.</u> $\vec{r}$ is <u>differentiable</u> if and only if its component functions are.

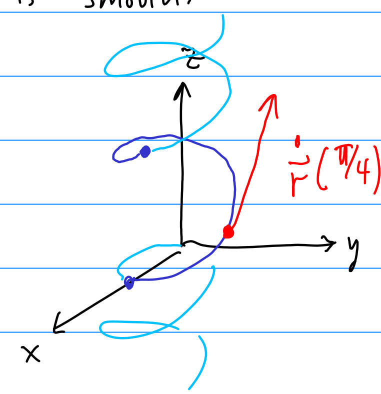
Geometrically



- $\dot{\vec{r}}$ is a vector tangent to the curve at the point.
- Length depends on the choice of parametrization.

Def'n. A vector function is smooth if $\dot{\vec{r}}$ exists at every point and $\dot{\vec{r}} \neq \vec{0}$. We also say that C is smooth.

Ex. $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ Helix



$\dot{\vec{r}}(t) = \langle \frac{d}{dt}\cos t, \frac{d}{dt}\sin t, \frac{d}{dt} t \rangle$
$= \langle -\sin t, \cos t, 1 \rangle$

$\dot{\vec{r}}(\pi/4) = \langle -\sin \pi/4, \cos \pi/4, 1 \rangle = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 \rangle$

$\|\dot{\vec{r}}\| =$ "speed" $= \sqrt{\underline{\sin^2 t + \cos^2 t} + 1^2} = \sqrt{2}$

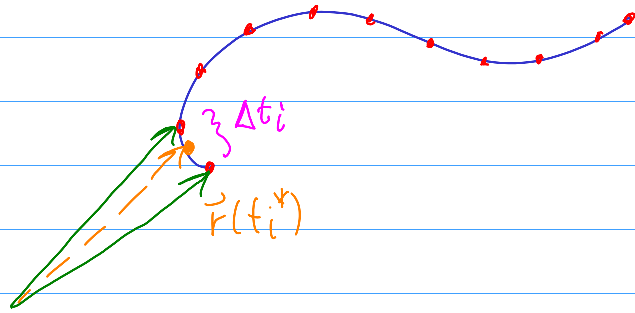
Ex. $\vec{r}(t) = \langle \cos(3t), \sin(3t), 3t \rangle$
$\dot{\vec{r}}(t) = \langle -3\sin(3t), 3\cos(3t), 3 \rangle$
$\|\dot{\vec{r}}(t)\| = \sqrt{9\sin^2(3t) + 9\cos^2(3t) + 9} = \sqrt{18} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$

Def'n. A vector function is continuous iff $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$.

"The function does what we expect it to do."

Def'n. Let $\vec{r}$ be a continuous vector function. The integral of $\vec{r}$ is the vector function defined by

$$\int_a^b \vec{r}(t)\,dt = \lim_{N \to \infty} \sum_{i=1}^{N} \vec{r}(t_i^*)\,\Delta t_i$$



Cor. $\int_a^b \vec{r}(t)\,dt = \int_a^b \langle x(t), y(t), z(t)\rangle\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$

Defn. An $\underbrace{\text{anti-derivative}}$ of $\vec{r}$ is a vector function $\vec{R}(t)$ satisfying $\dot{\vec{R}}(t) = \vec{r}(t)$.

FTC: $\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a)$.

Ex. A particle has velocity given by $\vec{v}(t) = \langle \frac{1}{1+t^2}, t^3, \ln t \rangle$ for $t \geq 1$. If the particle starts at the origin, what is its position function?

$\vec{r}(1) = \vec{0}$

write $\dot{\vec{r}}(t) = \vec{v}(t)$

so, $\vec{r}(t) = \int \vec{v}(t)\,dt = \left\langle \int \frac{1}{1+t^2}\,dt, \int t^3\,dt, \int \ln t\,dt \right\rangle$

$= \langle \arctan(t) + C_1, \frac{1}{4}t^4 + C_2, t\ln t - t + C_3 \rangle$

$= \langle \arctan(t), \frac{1}{4}t^4, t\ln t - t \rangle + \vec{C}$

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$\int \underbrace{\ln t}_{u}\, \underbrace{dt}_{dv} = uv - \int v\,du = t\ln t - \int t/t\,dt = t\ln t - t + C_3$

$u = \ln t$ $\quad dv = dt$

$du = 1/t\,dt$ $\quad v = t$

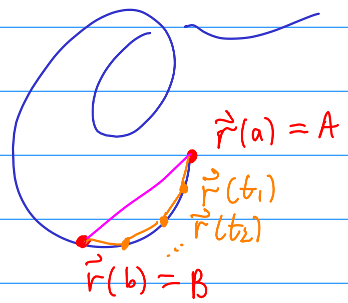
Initial condition: $\vec{r}(1) = \langle 0,0,0 \rangle$

Integral: $\vec{r}(t) = \langle \arctan t,\ \frac{1}{4}t^4,\ t\ln t - t \rangle + \vec{C}$

$$\vec{r}(1) = \langle \arctan(1),\ \tfrac{1}{4},\ -1 \rangle + \vec{C} = \langle 0,0,0 \rangle$$

$$\text{so} \quad \vec{C} = \langle -\pi/4,\ -1/4,\ 1 \rangle$$

$$\vec{r}(t) = \langle \arctan(t) - \pi/4,\ \tfrac{1}{4}t^4 - 1/4,\ t\ln t - t + 1 \rangle$$



Arc length of a curve segment?

$$\text{arc length} = s \approx |AB| = \underbrace{\|\vec{r}(b) - \vec{r}(a)\|}_{\Delta t}$$

$$\text{Better:} \quad s \approx \sum_{i=1}^{n} \underbrace{\|\vec{r}(t_i) - \vec{r}(t_{i-1})\|}_{\Delta t_i}$$

<u>Def'n.</u> The <u>arclength</u> of a segment of a space curve between $t=a$ and $t=b$ is given by

$$\boxed{s = \int_a^b \|\dot{\vec{r}}(t)\|\, dt}$$

"Distance = rate × time"

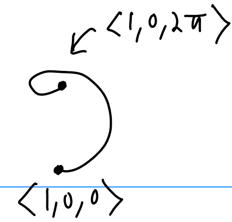
$s \qquad \|\dot{\vec{r}}\| \qquad \Delta t$

<u>Ex.</u> Find the arc length of one cycle of the helix.

$\vec{r}_1(t) = \langle \cos t, \sin t, t \rangle$ and $\underbrace{\vec{r}_2(t) = \langle \cos(3t), \sin(3t), 3t \rangle}$

$\dot{\vec{r}}_1(t) = \langle -\sin t, \cos t, 1 \rangle$ and $\|\dot{\vec{r}}_1(t)\| = \sqrt{2}$

$$s = \int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2}$$



$\dot{\vec{r}}_2(t) = \langle -3\sin(3t), 3\cos(3t), 3\rangle$ and $\|\dot{\vec{r}}_2\| = 3\sqrt{2}$

$$3t = 2\pi \rightarrow t = \frac{2}{3}\pi$$

then $s = \int_0^{2\pi/3} 3\sqrt{2}\, dt = \frac{2\pi}{\cancel{3}} \cdot \cancel{3}\sqrt{2} = 2\pi\sqrt{2}$

so arc length is <u>independent</u> of the parametrization.

<u>Ex.</u> $\vec{r}(t) = \langle t, t^2, t^3\rangle$ Write out the integral for the arc length from 0 to 1.

$\dot{\vec{r}}(t) = \langle 1, 2t, 3t^2\rangle$ so <u>smooth</u> for all t.

$\|\dot{\vec{r}}(t)\| = \sqrt{1+4t^2+9t^4}$

so $s = \int_0^1 \sqrt{1+4t^2+9t^4}\, dt$ Computer or numerical problem.