M344 - Calculus III 21 Jan 2020

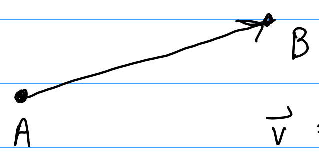
J. Stewart, Calculus, 8e
Ch 13–16

§13.1/16.6 – Space Curves and Vector Functions, Parametrized Surfaces

Def'n. A vector is a quantity w/ magnitude and direction.

Ex. velocity is a vector
speed is not

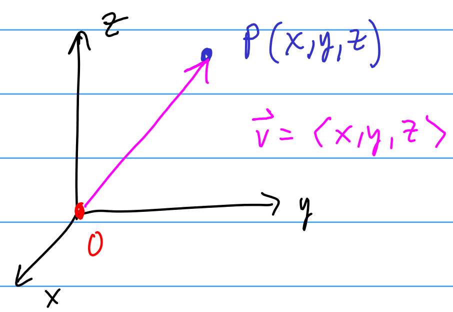
Think of vectors as "arrows" or directed line segment.



$\vec{v} = \overrightarrow{AB}$

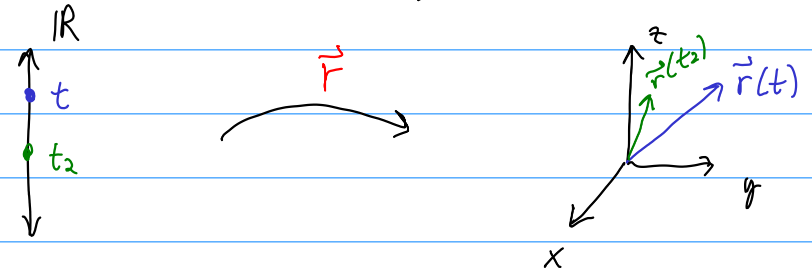
A = initial point
B = terminal point

A vector is said to be in standard form if its initial point is the origin. The terminal point then completely determines the vector.



Def'n. We will denote the space of all vectors in $\mathbb{R}^n$ by $\mathbb{V}^n$.
For us, $\mathbb{V}^2$ and $\mathbb{V}^3$

Defn. A vector function is a function whose domain is $\mathbb{R}$ and the range is V^n. $(n=2,3)$



we write $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Recall: If we regard x, y, z as individual functions of t, then $\vec{r}$ can be thought of as a parametric function:

$$\vec{r}(t) = \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

Ex. $\vec{r}(t) = \langle e^{2t}, \ln(t^2-1), \sin(4t) \rangle$

The domain of $\vec{r}$ is the intersection (overlap) of the ind. component functions: $\mathrm{dom}(\vec{r}) = \mathrm{dom}(x) \cap \mathrm{dom}(y) \cap \mathrm{dom}(z)$.

$x = e^{2t}$ $\quad$ $\mathrm{dom}(x) = \mathbb{R}$

$y = \ln(t^2-1)$ $\quad$ $t^2 - 1 > 0 \rightarrow t^2 > 1$ $\quad$ $(-\infty,-1) \cup (1,\infty)$

$z = \sin(4t)$ $\quad$ $\mathrm{dom}(z) = \mathbb{R}$

therefore, $\mathrm{dom}(\vec{r}) = (-\infty,-1) \cup (1,\infty)$

Defn. The limit of a vector function $\vec{r} = \langle x, y, z \rangle$ is

$$\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \right\rangle$$

provided the component limits all exist.

Ex. $\vec{r}(t) = \langle e^{2t}, \ln(t^2-1), \sin(4t) \rangle$

$$\lim_{t \to 1^+} \vec{r}(t)$$

x: $\displaystyle\lim_{t \to 1^+} e^{2t} = e^2$

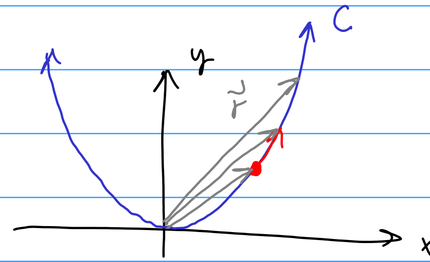
y: $\displaystyle\lim_{t \to 1^+} \ln(t^2-1) = \ln\left(\lim_{t \to 1^+} t^2-1\right) \simeq \ln(0^+) = \lim_{u \to 0^+} \ln(u) = -\infty$

z: $\displaystyle\lim_{t \to 1^+} \sin(4t) = \sin(4)$

$\displaystyle\lim_{t \to 1^+} \vec{r}(t) = \text{DNE}$ ∎

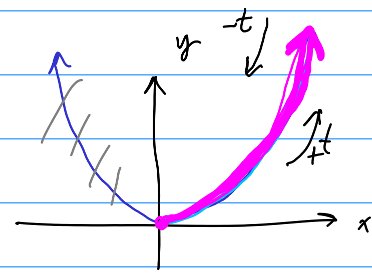
Defn. A space curve is the graph traced out by the terminal points of a vector function.
i.e., it's a parametrized curve.

Ex. $\vec{r}(t) = \langle t, t^2 \rangle$ $\qquad \text{dom}(\vec{r}) = \mathbb{R}$



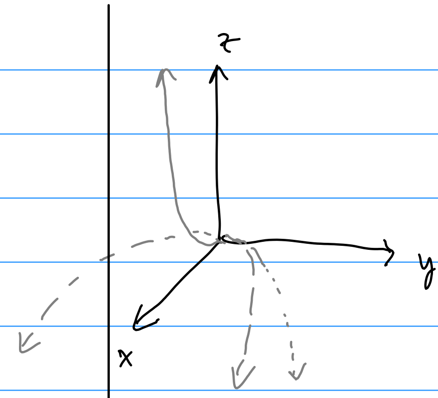
$x = t$
$y = t^2 = x^2$

Ex. $\vec{r}(t) = \langle t^2, t^4 \rangle$ $\qquad \text{dom}(\vec{r}) = \mathbb{R}$



$x = t^2$
$y = t^4 = (t^2)^2 = x^2$

Ex. $\vec{r}(t) = \langle t, t^2, t^3 \rangle$



xy-plane: parabola

xz-plane: cubic $z = x^3$

"Twisted cubic"

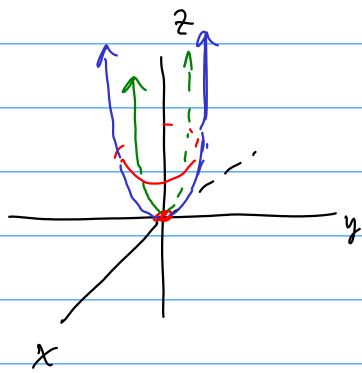
Ex. $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

16.6 – Param. Surfaces

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

Ex. $z = x^2 + y^2$

$\vec{r}(u,v) = \langle u, v, u^2 + v^2 \rangle$



Sphere?

$x^2 + y^2 + z^2 = 1 \rightarrow \quad z = \pm\sqrt{1 - x^2 - y^2}$