Last week we defined the moduli space and Teichmüller space of an arbitrary Riemann surface S, via the groups Homeo, Homeo_0, and MC.

Recall, $St_{\mathbb{C}}(S)$ is the set of all complex structures on S. Homeo(S) and $\text{Homeo}_0(S)$ act on $St_{\mathbb{C}}(S)$ by

$$\begin{aligned} \tau: St_{\mathbb{C}}(S) \times \text{Homeo}(S) &\to St_{\mathbb{C}}(S) \\ : \{\varphi\} \times f &\longmapsto \{\varphi \circ f\} \end{aligned}$$

We define $\mathcal{M}(S) = St_{\mathbb{C}}(S)/\text{Homeo}(S)$ and
$\mathcal{T}(S) = St_{\mathbb{C}}(S)/\text{Homeo}_0(S)$

The mapping class group is $MC(S) = \text{Homeo}(S)/\text{Homeo}_0(S)$. The moduli space and Teichmüller space are related by $\mathcal{M}(S) = \mathcal{T}(S)/MC(S)$.

Now let S be a closed (compact, no boundary) Riemann surface. We have already seen that S is completely (?) classified (at least topologically) by its genus — or equivalently its Euler characteristic. Thus the moduli and Teichmüller spaces of S depend only on the genus of S.

For a closed Riemann surface S of genus g, we define

$$\mathcal{M}_g = St_{\mathbb{C}}(S)/\text{Homeo}(S)$$
$$\mathcal{T}_g = St_{\mathbb{C}}(S)/\text{Homeo}_0(S).$$

There are other points of view and corresponding constructions of the Teichmüller spaces T_p.

First, following Jost, Ch. 4.

We've already discussed surfaces of genus 1 (tori), so assume S is a closed Riemann surface of genus $p \geq 2$. Thus S is covered by H and automatically inherits a hyperbolic metric.

The hyperbolic metric is uniquely determined by the conformal structure on S since any conformal map between hyperbolic metrics is an isometry: any such map lifts to a conformal automorphism of H, which is an isometry of g_H on H.

Thus, two conformally equivalent hyperbolic metrics differ only by an isometry, and we cannot distinguish between them.

<u>Lemma 4.1.1</u> Let S be a compact surface of genus $p \geq 2$. Then there is a natural bijective correspondence between conformal structures and hyperbolic metrics on S.

Just as we cannot distinguish between isometric metrics, we cannot distinguish between conformally equivalent conformal structures.

<u>Defn 4.1.1</u> The moduli space $\mathcal{M}_p$ is the set of all hyperbolic metrics on S, two metrics being identified iff they are isometric.

As we stated before, the topology of $\mathcal{M}_p$ is in general quite complicated. Thus, we introduce the Teichmüller space, $\mathcal{T}_p$.

We identify (S, g_1) and (S, g_2) iff there exists an isometry between them, $f: S \to S$ that is homotopic (isotopic!) to the identity map. (Recall: isometries live in $\mathrm{Diff}(S)$, and isotopic means the homotopy lives entirely in $\mathrm{Diff}(S)$.)

Now consider the triple (S, g, f) where g is a hyperbolic metric and $f: S \to S$ is a diffeomorphism. Two triples (S, g_1, f_1) and (S, g_2, f_2) will be considered equivalent iff there exists a conformal map $k: (S, g_1) \to (S, g_2)$ for which the diagram commutes up to homotopy.

$$\begin{tikzcd} & (S, g_1) \arrow[dd, "k"] \\ S \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\ & (S, g_2) \end{tikzcd}$$

i.e., $k \sim f_2 \circ f_1^{-1}$ are homotopic.

<u>Def'n 4.1.2</u> The space of equivalence classes of triples (S, g, f) under the above relation is called Teichmüller space and is denoted by $\mathcal{T}_p$, $p = \text{genus}(S)$.

The map $f \in \mathrm{Diff}(S)$ is called a marking of S. Thus, the moduli space $\mathcal{M}_p$ is the quotient of $\mathcal{T}_p$ obtained by "forgetting" the marking.

Thus, f tells us how (S,g) is identified with the topological model S. Thus, (S, g_1, f_1) and (S, g_2, f_2) are identified iff there exists a conformal diffeomorphism $f:(S,g_1) \to (S,g_2)$ that is isotopic to the identity.

Yet another point-of-view, due to [Imayoshi–Taniguchi].

Let R be a closed Riemann surface. A <u>marking</u> of R is a system of canonical generators $\{[A_j],[B_j]\}_{j=1}^{g}$ of a fundamental group $\pi_1(R,p)$, where $g = \text{genus}(R)$.

Two markings are (Teichmüller) equivalent iff there exists a continuous curve C_0 on R such that $[A_j'] = T_{C_0}([A_j])$ and $[B_j'] = T_{C_0}([B_j])$ for $j=1,\dots,g$, where T_{C_0} is the isomorphism of $\pi_1(R,p)$ to $\pi_1(R,p')$ sending any $[C]$ to $[C_0^{-1}\cdot C \cdot C_0]$.

Let Σ_p and Σ_q be markings on closed Riemann surfaces R and S of genus g. The pairs (R,Σ_p) and (S,Σ_q) are said to be equivalent iff there exists a biholomorphic (conformal) map $h: S \to R$ such that $(h_* \Sigma_q)$ is equivalent to Σ_p. The equivalence class $[R,\Sigma_p]$ is called a <u>marked Riemann surface</u> and the Teichmüller space T_g is the space of all marked Riemann surfaces of genus g.

Yet another equivalent point-of-view—

Fix a closed Riemann surface R, of genus g. Consider an arbitrary pair (S, f) where S is a closed Riemann surface and $f: R \to S$ is an orientation-preserving diffeomorphism. Two pairs (S, f_1) and (S', f_2) are said to be equivalent iff $f_2 \circ f_1^{-1}: S \to S'$ is homotopic to a biholomorphic map $h: S \to S'$. The Teichmüller space $\mathcal{T}(R)$ is the space of all equivalence classes.
Here, f is the marking.

RE. Show that these last two are equivalent.

Clearly the last one corresponds to our original definition.

Notation & Vocab.
Teichmüller theorists call the mapping class group $MC(S)$ the Teichmüller modular group. The elements are called Teichmüller modular transformations.

Recalling our discussion of the Teichmüller space $\mathcal{T}_1$, this is because of the relationship between $MC(T^2)$ and the modular group $SL(2, \mathbb{Z})$.

Future— We will see that $\mathcal{T}(R)$ is a $(3g-3)$-dim complex manifold and $MC(R)$ acts properly discontinuously on $\mathcal{T}(R)$.

Still following [IT],

<u>Quasiconformal mappings and Beltrami coefficients.</u>

Consider a fixed closed Riemann surface R and a point $[S,f] \in T(R)$. We want to compare the complex structures on R and S.

Take a coordinate neighborhood (U,z) on R and (V,w) on S with $f(U) \subset V$, and set $F := w \circ f \circ z^{-1}$. Then

$$\mu = \frac{\partial_{\bar z} F}{\partial_z F} \qquad (*)$$

is a smooth complex-valued function defined on an open set $z(U)$ in the complex plane.

It is independent of the choice of w.

Since f is orientation-preserving, the Jacobian of F

$$DF = |\partial_z F|^2 - |\partial_{\bar z} F|^2$$

is positive definite on $z(U)$. Thus $|\mu| < 1$ on $z(U)$.

Moreover, F is biholomorphic on $z(U)$ if and only if $\mu = 0$ on $z(U)$.

μ is the <u>Beltrami</u> <u>coefficient</u> of f wrt (U, z).

<u>Note</u>: The Beltrami coefficient is <u>local</u>! It depends on the choice of chart (U, z) on R.

Let (U_j, z_j), (U_k, z_k) be coordinate charts on R, and (V_j, w_j), (V_k, w_k) charts on S such that $f(U_j) \subset V_j$ and $f(V_k) \subset U_k$. Let μ_j and μ_k be the Beltrami coefficients. When $U_j \cap U_k \neq \emptyset$,

$$\mu_j = (\mu_k \circ z_{kj}) \frac{\overline{\left(\frac{dz_{kj}}{dz_j}\right)}}{\left(\frac{dz_{kj}}{dz_j}\right)} \quad \text{on} \quad z_j(U_j \cap U_k)$$

where $z_{kj} = z_k \circ z_j^{-1}$.

Thus, we may write μ globally as

$$\mu_f = \mu \frac{d\bar{z}}{dz} \qquad (**)$$

and call this the (global) <u>Beltrami</u> <u>coefficient</u> on R.

Now for a system of local coordinates $\{(V_\alpha, w_\alpha)\}$ on S an an orientation preserving diffeomorphism $f: R \to S$, a system of coordinates $\{(f^{-1}(V_\alpha), w_\alpha \circ f)\}$ defines a complex structure on R. In this way, we obtain a new Riemann surface R_f equipped w/ coordinate nbds $\{(f^{-1}(V_\alpha), w_\alpha \circ f)\}$.

Note, $R_f = R$ as sets and $id: R_f \to R$ is an orientation-preserving diffeomorphism. Moreover, $f: R_f \to S$ is a biholomorphism.

Thus $[S, f] = [R_f, id]$ in $\mathcal{T}(R)$.

Therefore a point $[S, f]$ in $\mathcal{T}(R)$ represents a deformation of the complex structure on R.

The Beltrami coefficient measures the deviation of f from conformality.

We will study this further.