## §2.7. Conformal Structures on Tori

Let $f$ be a meromorphic function on $\mathbb{C}$. (Recall, a function is meromorphic on $\mathbb{C}$ if it is holomorphic on $\mathbb{C}\setminus P$ where $P$ is a set of isolated points — called poles.)

An $\omega \in \mathbb{C}$ is said to be a period of $f$ iff

$$f(z+\omega) = f(z) \qquad (*)$$

for all $z \in \mathbb{C}$.

The periods of $f$ form a module over $\mathbb{Z}$, call it $M$. (in fact, an additive subgroup.)

If $f$ is non-constant, then $M$ is discrete.

The possible discrete subgroups of $\mathbb{C}$ are

$$M = \{0\}$$

$$M = \{n\omega \mid n \in \mathbb{Z}\}$$

(*) $\longrightarrow$ $M = \{m\omega_1 + n\omega_2 \mid m,n \in \mathbb{Z}\}, \quad \frac{\omega_2}{\omega_1} \notin \mathbb{R}$

Of these, the third case is the most interesting because it is a lattice in $\mathbb{C}$.

As we know, such a lattice defines a torus $T = T_M$ if we identify points $z \sim z + m\omega_1 + n\omega_2$ and let $\pi : T_M \to \mathbb{C}$ be the projection as before.

The parallelogram determined by $\omega_1$ and $\omega_2$ is a fundamental domain for $T$.

By $(*)$, $f$ becomes a meromorphic function on $T_M$.

If $(\omega_1', \omega_2')$ is another basis for the same module, the change-of-basis is described by

$$\begin{pmatrix} \omega_2' & \overline{\omega_2'} \\ \omega_1' & \overline{\omega_1'} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 & \overline{\omega_2} \\ \omega_1 & \overline{\omega_1} \end{pmatrix}$$

with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belonging to

$$GL(2, \mathbb{Z}) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\middle|\; A, B, C, D \in \mathbb{Z} \text{ and } \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \pm 1 \right\}.$$

Its subgroup $SL(2, \mathbb{R})$ is called the modular group, and elements of $SL(2, \mathbb{R})$ are called unimodular transformations.

$PSL(2, \mathbb{Z}) := SL(2, \mathbb{R}) / \{\pm Id\}$ is a subgroup of $PSL(2, \mathbb{R})$, hence acts on $\mathbb{H}$ by isometries.

**Thm 2.7.1** There is a basis $(\omega_1, \omega_2)$ for $M$ such that for $\tau := \frac{\omega_2}{\omega_1}$,

i) $\operatorname{Im} \tau > 0$,

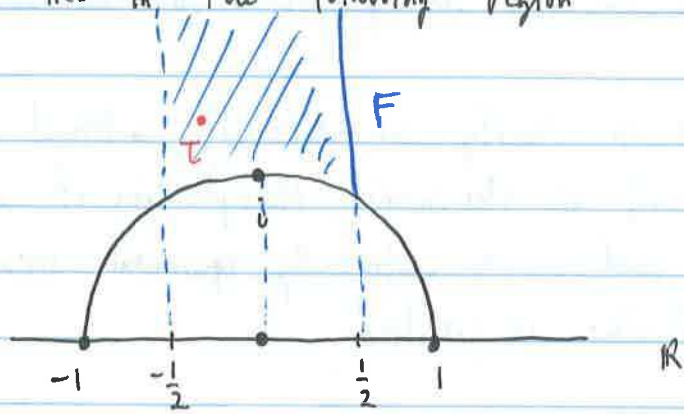
ii) $-\frac{1}{2} < \operatorname{Re} \tau \leq \frac{1}{2}$,

iii) $|\tau| \geq 1$,

iv) $\operatorname{Re} \tau \geq 0$ if $|\tau| = 1$.

$\tau$ is uniquely determined by these conditions, and the number of such bases for a given module $M$ is 2, 4, or 6.

That is, $\tau$ lies in the following region




This region is a fundamental domain (polygon, in fact) for the action of $PSL(2,\mathbb{Z})$ on $\mathbb{H}$.

That there are in general 2 bases for a given $\tau$ follows from replacing $(-w_1,-w_2)$ for $(w_1,w_2)$.

If $\tau = i$, then there are 4 bases, namely we can also replace $(w_1,w_2)$ by $(iw_1, iw_2)$.

Finally, we get 6 bases when $\tau = e^{i\frac{\pi}{3}}$. In this case we can replace $(w_1,w_2)$ by $(\tau w_1, \tau w_2)$ and $(\tau^2 w_1, \tau^2 w_2)$.

Note: $\tau = i$ and $\tau = e^{i\frac{\pi}{3}}$ are precisely the fixed points of non-trivial elements of $PSL(2,\mathbb{Z})$ in $\bar{F}$.

The normalisation of Thm 2.7.1 may also be interpreted as: choose $w_1 = 1$, then $w_2$ lies in F.

We shall use this last perspective in the sequel.

We want to classify the different conformal classes of tori (they're not all the same!). Multiplication of a module basis will always lead to a conformally equivalent torus. Hence the latter point-of-view is justified.

One again we mention the Uniformization Theorem: Every Riemann surface that is homeomorphic to a torus is in fact conformally equivalent to a quotient of $\mathbb{C}$, $T_M = \mathbb{C}/M$. (The universal cover of such a surface is $\mathbb{C}$.)

Recall from §1.3, $\pi_1(T_M) = \mathbb{Z} \oplus \mathbb{Z}$ since the group of covering transformations of $\pi : \mathbb{C} \to T_M$ is $\mathbb{Z} \oplus \mathbb{Z}$, generated by the maps

$$\varphi_1 : z \longmapsto z + w_1$$
$$\varphi_2 : z \longmapsto z + w_2$$

Thus, the fundamental group $\pi_1(T_M)$ is canonically isomorphic to the module $M = \{mw_1 + nw_2 \mid m, n \in \mathbb{Z}\}$.

**Lemma 2.7.1** Let $f_1, f_2 : T_M \to T_{M'}$ be continuous maps between tori. Then $f_1$ and $f_2$ are homotopic if and only if the induced maps

$$f_{i*} : \pi_1(T_M) \to \pi_1(T_{M'}) \qquad i = 1,2$$

coincide.

Assume the induced maps coincide.

Proof. Consider lifts $\tilde{f}_i : \mathbb{C} \to \mathbb{C}$. Let $w_1, w_2$ be a basis of $\pi_1(T_M)$. Then by assumption

$$\tilde{f}_1(z + mw_1 + nw_2) - \tilde{f}_2(z + mw_1 + nw_2) = \tilde{f}_1(z) - \tilde{f}_2(z)$$

for all $z \in \mathbb{C}$, $m, n \in \mathbb{N}$. It follows that $\tilde{F}(z,s) := (1-s)\tilde{f}_1(z) + s\tilde{f}_2(z)$ satisfies

$$\tilde{F}(z+mw_1+nw_2, s) = \tilde{f}_1(z+mw_1+nw_2) + s(\tilde{f}_2(z) - \tilde{f}_1(z)).$$

Hence each $\tilde{F}(\cdot, s)$ induces a map $F(\cdot, s) : T_M \to T_{M'}$. This provides the homotopy between $f_1$ and $f_2$.

The other direction follows from Lemma 1.2.3 applied to a homotopy between $f_1$ and $f_2$. $\square$

The discussion to follow will provide a simple case of some concepts that we will discuss in much more generality later.

Again, we assume that a torus $T_M = T_\tau$ is generated by a module $M$ w/ basis $\{1, \tau\}$, $\tau \in F$ of our previous discussion.

Def'n 2.7.1 The moduli space $M_1$ is the space of equivalence classes of tori, two tori being regarded as equivalent iff there exists a bijective conformal map between them.

We say that a sequence of equivalence classes, represented by Tori $T^n$, $n \in \mathbb{N}$, converges to the equivalence class of $T$ if we can find bases $(w_1^n, w_2^n)$ for $T^n$ and $(w_1, w_2)$ for $T$ s.t. $\tau^n \to \tau$.

Def'n 2.7.2 The Teichmüller space $\mathfrak{T}_1$ ("T") is the space of equivalence classes of pairs $(T, (w_1, w_2))$ where $T$ is a torus and $(w_1, w_2)$ is a basis of the module defining $T$. Here, $(T, (w_1, w_2))$ and $(T', (w_1', w_2'))$ are equivalent iff there exists a bijective conformal map

$$f: T \to T'$$

w/ $f_*(w_i) = w_i'$.

Note: Here $(w_1, w_2)$ and $(w_1', w_2')$ have been canonically identified w/ bases of $\pi_1(T)$ and $\pi_1(T')$, and $f_*$ is the induced map of homotopy classes.

We say that $(T^n, (w_1^n, w_2^n)) \to (T, (w_1, w_2))$ iff $\tau^n \to \tau$.

The pair $(T, (w_1, w_2))$ is called a marked torus.

Remarks Consider a marking $(w_1, w_2) = (1, i)$ (so $\tau = i$) and let $T_i = T_{mod}$ be the model topological space. By Lemma 2.7.2, $(w_1, w_2)$ defines a homotopy class $\alpha(w_1, w_2)$ of maps $T_\tau \to T_{mod}$, where $\tau = \frac{w_2}{w_1}$.

Namely, $\alpha(w_1, w_2)$ is that homotopy class for which the induced map of fundamental groups sends $(w_1, w_2)$ to the given basis of $T_{mod}$ $(w_1 \mapsto 1, w_2 \mapsto i)$.

Such a map always exists: take the $\mathbb{R}$-linear map $g: \mathbb{C} \to \mathbb{C}$ w/ $g(w_1) = 1$ and $g(w_2) = i$.

Now, instead of pairs $(T, (w_1, w_2))$, we could consider pairs $(T, \alpha)$ where $\alpha$ is a homotopy class of maps $T \to T_{mod}$ which induces an isomorphism of fundamental groups (thus $\alpha$ should contain a homeomorphism). $(T, \alpha)$ and $(T', \alpha')$ are now regarded as equivalent iff the homotopy class $(\alpha')^{-1} \circ \alpha : T \to T'$ contains a conformal map. $\mathcal{T}_1$ is then the space of such equivalence classes.

<u>Thm. 2.7.2.</u> $\mathcal{T}_1 = H$ and $\mathcal{M}_1 = PSL(2,\mathbb{Z}) \backslash H$. (left action quotient)

We have already seen that every torus is conformally equivalent to a $T_\tau$ w/ $\tau \in F$.

Similarly, every marked torus can be identified w/ an element of $H$ — just normalise so that $w_1 = 1$.

Thus we must show that two distinct elements of $PSL(2,\mathbb{Z}) \backslash H$ (resp. $H$) are not conformally equivalent (resp. equivalent as marked tori). This is task of the remainder of Ch. 2.

<u>Def'n 2.7.3.</u> A map $h : T \to T'$ is said to be <u>harmonic</u> iff its lift $\tilde{h} : \mathbb{C} \to \mathbb{C}$ is harmonic.

Equivalently, every local representation of $h$ should be harmonic, but this requires us to check that the harmonicity is preserved under change-of-charts.

<u>Lemma 2.7.2</u> Let $T, T'$ be tori, $z_0 \in T$, $z_0' \in T'$. Then in every homotopy class of maps $T \to T'$, there exists a harmonic map $h$; $h$ is uniquely determined by requiring that $h(z_0) = z_0'$. The lift $\tilde{h} : \mathbb{C} \to \mathbb{C}$ is affine $\mathbb{R}$-linear.
If normalised by $\tilde{h}(0) = 0$ (instead of $h(z_0) = z_0'$), it is linear.
$h$ is conformal iff the normalised $\tilde{h}$ is of the form $z \mapsto \lambda z$, $\lambda \in \mathbb{C}$.

<u>Proof.</u> Let $(w_1, w_2)$ be a "basis" of $T$ and $(w_1', w_2')$ the image determined by the given homotopy class. Then the $\mathbb{R}$-linear map $\tilde{h} : \mathbb{C} \to \mathbb{C}$ w/ $h(w_i) = w_i'$ induces a harmonic map $h : T \to T'$ in the given homotopy class.

Now suppose $\tilde{f}$ is the lift of any map $f : T \to T'$ in the homotopy class. Then

$$\tilde{f}(z + mw_1 + nw_2) = \tilde{f}(z) + mw_1' + nw_2',$$

hence

$$\frac{\partial \tilde{f}}{\partial x}(z + mw_1 + nw_2) = \frac{\partial \tilde{f}}{\partial x}(z)$$

and similarly for $\frac{\partial \tilde{f}}{\partial y}$. Thus if $f$ (hence also $\tilde{f}$) is harmonic, then so are $\partial_x \tilde{f}$ and $\partial_y \tilde{f}$.

But $\partial_x \tilde{f}, \partial_y \tilde{f}$ are $\mathbb{C}$-valued harmonic functions on $T$, hence constant by Lemma 2.2.1.

Thus $\tilde{f}$ is "affine" $\mathbb{R}$-linear. It also follows that the harmonic map in a homotopy class is uniquely determined by $h(z_0) = z_0'$.

From another pt-of-view, the difference between the lifts of two harmonic maps becomes a harmonic function on $T$, and is therefore constant. $\square$

The proof of the theorem is now immediate.

A conformal map is harmonic, hence has a harmonic lift $\tilde{h}$ by Lemma 2.7.2. WLOG assume $\tilde{h}(0) = 0$.
Also assume the markings are normalized by $w_1 = 1 = w_1' \ (= \tilde{h}(w_1))$.
But if $\tilde{h}(1) = 1$, then $\tilde{h}$ is the identity map (by the Lemma).
Thus, $\mathcal{T}_1 = \mathbb{H}$.

For $\mathcal{M}_1$, choose any markings $T = T_\tau$ and $T' = T_{\tau'}$. $T_\tau$ and $T_{\tau'}$ are conformally equivalent precisely when $(1, \tau')$ is a basis for $T_\tau$. But $T_\tau$ is generated by $(1, \tau)$. Hence $\tau' = \tau$.
Thus $\mathcal{M}_1 = PSL(2,\mathbb{Z}) \backslash \mathbb{H}$. $\square$

---

Let $S$ be a Riemann surface. $\mathrm{Homeo}(S)$ is the group of all homeomorphisms $f: S \to S$ with composition as the operation.
Let $f, g \in \mathrm{Homeo}(S)$. We say $f$ is isotopic to $g$ iff there exists a homotopy between $f$ and $g$ in $\mathrm{Homeo}(S)$.

The open sets in Homeo$(S)$ consist of maps that send compact subsets $K$ into open subsets $U$ as $K$ and $U$ vary in the topology of $S$, completed w/ their finite intersections and arbitrary unions. (Compact-open topology).

Let Homeo$_0(S)$ be the connected component of the identity map. This consists of all homeomorphisms isotopic to the identity. Homeo$_0(S)$ is normal in Homeo$(S)$.

The mapping class group of $S$ is

$$MC(S) = MC_S = \text{Homeo}(S)/\text{Homeo}_0(S)$$

It consists of all isotopy classes of homeomorphisms.

Now, Homeo$(S)$ and Homeo$_0(S)$ act naturally on $S$. Let $St_{\mathbb{C}}(S)$ denote the set of complex structures ($\{\varphi_\alpha : U_\alpha \to \mathbb{C}\}$ maximal atlas) on $S$. Denote the elements of $St_{\mathbb{C}}(S)$ by $\{\varphi\}$. Then

$$\tau : \text{Homeo}(S) \times St_{\mathbb{C}}(S) \to St_{\mathbb{C}}(S) : (f, \{\varphi\}) \to \{\varphi \circ f\}$$

is a (right) action of Homeo$(S)$ on $St_{\mathbb{C}}(S)$.

We now define,

$$\mathcal{M}_1 = St_{\mathbb{C}}(T^2)/\text{Homeo}(T^2)$$

and

$$\mathcal{T}_1 = St_{\mathbb{C}}(T^2)/\text{Homeo}_0(T^2)$$

Notice that $\mathcal{M}_1 = \mathcal{T}_1/MC(T^2)$ as well.

Another proof of Riemann–Hurwitz:

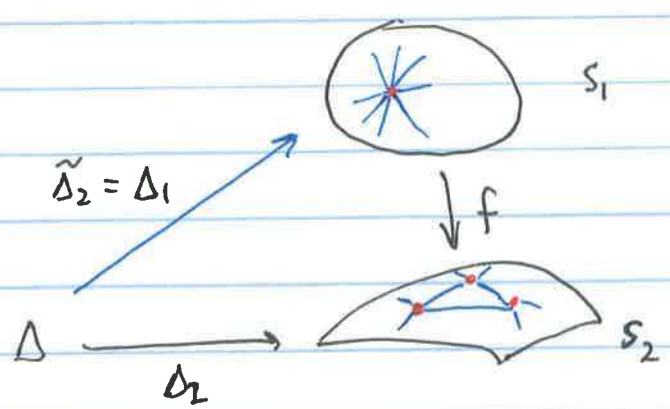
Recall,

<u>Thm (Riemann–Hurwitz)</u>
Let $f: S_1 \to S_2$ be a non-constant holomorphic map between closed (compact/no boundary) Riemann surfaces of degree $m$ and w/ total ramification $\nu_f$. Then

$$\chi(S_1) = m\,\chi(S_2) - \nu_f .$$

<u>Sketch of proof.</u>

Suppose $\Delta_2$ is a triangulation of $S_2$ for which all branch points are vertices. (There may be other vertices.)




Each edge and face in $\Delta_2$ lifts to exactly $m$ edges or faces in $\Delta_1$. If a vertex is not a branch point in $\Delta_2$, then it also lifts to exactly $m$ vertices.

However, a vertex in $S_2$ that is a branch point lifts to

$$m - \sum_{p \in f^{-1}(q)} N_f(p)$$

vertices in $S_1$. Thus, the formula in the Riemann-Hurwitz formula follows from the formula for the Euler characteristic in each space.