We now turn our attention to the Schwarz Lemma.

Thm 2.6.1 (Ahlfors-Schwarz Lemma)
Consider the disk D w/ hyperbolic metric $g_D = \lambda^2(z)\,dz\,d\bar{z} = \frac{4}{(1-|z|^2)^2}\,dz\,d\bar{z}$.
Let S be a Riemann surface w/ metric $g_\rho = \rho^2(z)\,dz\,d\bar{z}$ whose curvature satisfies $K_\rho \le -\varkappa < 0$ for some constant $\varkappa > 0$. Then for any holomorphic map $f: D \to S$ we have

$$\rho^2(f(z))\,\frac{\partial f}{\partial z}\,\frac{\partial \bar{f}}{\partial \bar{z}} \le \frac{1}{\varkappa}\,\lambda^2(z).$$

Proof. Recall the curvature formulas

$$K_D = K_H = -\frac{4}{\lambda^2}\frac{\partial^2}{\partial z\,\partial\bar{z}}\log\lambda = -1$$

and

$$K_\rho = -\frac{4}{\rho^2(f(z))\,f_z\,\bar{f}_{\bar{z}}}\,\frac{\partial^2}{\partial z\,\partial \bar{z}}\log\left(\rho^2(f(z))\,f_z\,\bar{f}_{\bar{z}}\right)^{\frac{1}{2}} \le -\varkappa,$$

where $f_z = \frac{\partial f}{\partial z}$ and $\bar{f}_{\bar{z}} = \frac{\partial \bar{f}}{\partial \bar{z}}$, at all pts where $f_z \neq 0$, by Lemma 2.3.7 and the assumption $K_\rho \le -\varkappa < 0$.

Put $u := \frac{1}{2}\log\left(\rho^2(f(z))\,f_z\,\bar{f}_{\bar{z}}\right)$ so that $4\frac{\partial^2}{\partial z\,\partial\bar{z}}u \ge \varkappa e^{2u}$.
wherever u is defined; i.e., $f_z \neq 0$.

For any $0 < R < 1$, we also put

$$v_R(z) := \log\left(\frac{2R}{\varkappa^{1/2}(R^2-|z|^2)}\right), \qquad |z| < R$$

and compute

$$4\frac{\partial^2}{\partial z\,\partial\bar{z}}\,v_R = \varkappa\,e^{2v_R}.$$

Combining these, we get $\quad 4\frac{\partial^2}{\partial z\,\partial\bar{z}}(u - v_R) \ge \varkappa\left(e^{2u} - e^{2v_R}\right) \quad$ where $f_z \neq 0$. $\quad (*)$

Let $\Sigma := \{ |z| < R \mid u(z) > v_R(z) \}$.

Since $u \to -\infty$ as $f_z \to 0$, Σ cannot contain any zeros of f.
Hence (*) is valid in Σ.
By the maximum principal, $u - v_R$ cannot attain an interior maximum in Σ.
But the boundary of Σ (in $\mathbb{C}$) is contained in $|z| < R$, since $v_R(z) \to -\infty$ as $|z| \to R$.
Hence $u - v_R = 0$. on $\partial\Sigma$ by continuity.
This means that the maximum of $u - v_R$ (which must be in $\partial\Sigma$) is zero. That is, Σ must be empty.

We conclude, $u(z) \leq v_R(z)$, $|z| < R$.
Letting $R \to 1$ we get $u(z) \leq \log\left(\frac{2}{\kappa^{1/2}(1-|z|^2)}\right)$
which is equivalent to the result. □

Remarks. Thms 2.3.1 and 2.3.2 are special cases of this Schwarz Lemma.

The Schwarz Lemma shows the importance of negatively curved metrics on Riemann surfaces.
The remainder of this section exploits the strong connection between the conformal structure of a Riemann surface and the curvature properties of the metrics that can be put on it.

In particular, one can often construct a metric w/ suitable properties on a Riemann surface and deduce consequences for the holomorphic structure of the surface.

To study these properties, we define for any p, q in a Riemann surface S,

(**) $$d_H(p,q) := \inf\left\{ \sum_{i=1}^{n} d(z_i, w_i) \;\middle|\; n \in \mathbb{N},\ p_0, p_1, \dots, p_n \in S \text{ w/ } p_0 = p,\ p_n = q,\ f_i : D \to S \text{ holomorphic}, \ f_i(z_i) = p_{i-1},\ f_i(w_i) = p_i \right\}$$

where $d(\cdot,\cdot)$ is the distance function on D defined by the hyperbolic metric.

d_H satisfies the triangle inequality,

$$d_H(p,q) \le d_H(p,r) + d_H(r,q) \qquad p, q, r \in S.$$

d_H is also symmetric and non-negative. Hence,

Def'n 2.6.1 S is said to be hyperbolic iff d_H defines a distance function on S.
That is, iff $d_H(p,q) > 0$, $\forall p \neq q \in S$.

Note. This usage of "hyperbolic" is not standard, but is used in higher-dimensional geometry. (By "not standard", I mean not the same as the rest of the book.)

Remark. d_H is continuous in q for fixed $p \in S$, and if S is hyperbolic the topology on S defined by d_H is the same as the original one. If d_H is complete, then bounded sets are relatively compact.

Cor 2.6.1 Suppose S carries a metric $\rho^2(w)\,dw\,d\bar{w}$ with curvature K_ρ bounded above by a negative constant. Then S is hyperbolic (in the sense of Def'n 2.6.1).

Proof Let $p,q \in S$, $f: D \to S$ holomorphic w/ $f(z_1)=p$, $f(z_2)=q$ for some $z_1, z_2 \in D$. Let γ be the geodesic arc in D joining z_1 to z_2. Then

$$
\begin{aligned}
d(z_1,z_2) &= \int_\gamma \lambda(z)\,|dz| \\
&\geq C\int_\gamma \rho(f(z))\,|f_z|\,|dz| \qquad \text{by the Thm.} \\
&= C\int_{f(\gamma)} \rho(w)\,|dw| \\
&\geq C\, d_\rho(p,q)
\end{aligned}
$$

where $C>0$ is a constant and d_ρ is the distance on S defined by the metric $\rho^2(w)\,dw\,d\bar{w}$. The corollary follows. □

Remark. The proof of Cor 2.6.1 illustrates that under the assumptions of Thm 2.6.1, any holomorphic map $f: D \to S$ is distance-decreasing. This is essentially the content of Def'n 2.6.1.

Examples 1.) On the unit disk D, d_H coincides with the distance function defined by the hyperbolic metric, as a consequence of the Schwarz Lemma. (As it better!)

2.) $\mathbb{C}$ is not hyperbolic.
If $p, q \in \mathbb{C}$, $p \neq q$, $\exists$ holomorphisms $f_n : D \to \mathbb{C}$ with $f_n(0) = p$ and $f_n(\frac{1}{n}) = q$ $(n \in \mathbb{N})$. Hence $d_H(p,q) = 0$.
In view of Corollary 2.6.1, it follows that $\mathbb{C}$ cannot carry any metric w/ curvature bounded above by a negative constant.

Thus, the conformal structure puts restrictions on the possible metrics on a Riemann surface, even in the non-compact case.

Lemma 2.6.1 d_H is non-increasing under holomorphic maps:
If $h : S_1 \to S_2$ is a holomorphic map, then
$$d_H(h(p), h(q)) \leq d_H(p,q)$$
for all $p, q \in S_1$. In particular, d_H is invariant under biholomorphic maps:
$$d_H(h(p), h(q)) = d_H(p,q)$$
for all $p, q \in S_1$ if h is bijective and holomorphic.

Lemma 2.6.2 Let S be a Riemann surface and $\tilde{S}$ its universal covering. Then S is hyperbolic if and only if $\tilde{S}$ is.

Proof: Read it.

Thm 2.6.2. Let S_1, S_2 be Riemann surfaces, and $z_0 \in S_1$. Assume that S_2 is hyperbolic in the sense of Defn 2.6.1 and complete wrt d_H. Then any bounded holomorphic map $f : S_1 \setminus \{z_0\} \to S_2$ extends to a holomorphic map $\bar{f} : S_1 \to S_2$.

Lemma 2.6.3 Let $f: D\setminus\{0\} \to \mathbb{R}$ be a bounded harmonic function. Then f can be extended to a harmonic function on D.

Thm 2.6.3 If S is hyperbolic, then any holomorphic map $f: \mathbb{C} \to S$ is constant.

Proof. This follows from $d_H \equiv 0$ on $\mathbb{C}$. □

Corollary 2.6.2 An entire holomorphic function omitting two values is constant.

Thm 2.6.4 Let $\bar{S}$ be a compact surface, and $S := \bar{S} \setminus \{w_1, \dots, w_k\}$ for a finite number of points in $\bar{S}$. Assume that S is hyperbolic. Let Σ be a Riemann surface, and $z_0 \in \Sigma$. Then any holomorphic map $f: \Sigma\setminus\{z_0\} \to S$ extends to a holomorphic map $\bar{F}: \Sigma \to \bar{S}$.

And the "big" Picard theorem,

Cor 2.6.3 Let f be holomorphic in the punctured disk $0<|z|<R$, with an essential singularity at $z=0$. Then there is at most one value a for which $f(z)=a$ has only finitely many solutions in $0<|z|<R$.