To begin this week, we prove some major theorems regarding Riemann surfaces.

First, we have

<u>Thm 2.5.1</u> Let B be a hyperbolic triangle in $\mathbb{H}$ (so that the sides of B are geodesic arcs) w/ interior angles $\alpha_1, \alpha_2, \alpha_3$. Let $K_{\mathbb{H}}$ be the curvature of the hyperbolic metric (so $K_{\mathbb{H}} \equiv -1$). Then

$$\int_B K_{\mathbb{H}}\, dA_{\mathbb{H}} = \int_B \frac{1}{2i}\frac{1}{y^2}\, dz \wedge d\bar{z} = \sum_{i=1}^{3} \alpha_i - \pi.$$

<u>Proof</u>. In general,

$$\int_B K_\lambda \lambda^2 \frac{i}{2}\, dz \wedge d\bar{z} = -\int_B \frac{4\partial^2}{\partial z \partial \bar{z}} \log(\lambda) \frac{i}{2}\, dz \wedge d\bar{z}$$

$$= -\int_{\partial B} \frac{\partial}{\partial n} \log(\lambda)\, |dz|$$

by Stokes's theorem, where $\partial/\partial n$ is the outward unit normal vector field along the boundary ∂B.

Hence, for us

$$\int_B K_{\mathbb{H}} \frac{1}{y^2} \frac{i}{2}\, dz \wedge d\bar{z} = -\int_{\partial B} \frac{\partial}{\partial n} \log\left(\frac{1}{y}\right) |dz|.$$

Now, ∂B consists of 3 geodesic arcs a_1, a_2, a_3. So each a_i is either a Euclidean line perpendicular to the real axis, or a segment of a circle centered on the real axis.

In the former case, $\frac{\partial}{\partial n} \log y = 0$.

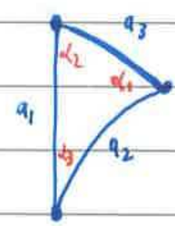
In the latter, we can write $y = r\sin\varphi$ in polar coords so that the integral becomes

$$-\int_{a_i} \frac{\partial}{\partial n}\log\frac{1}{y}|dz| = -\int_{\varphi_1}^{\varphi_2} \frac{\partial}{\partial r}\log\frac{1}{r\sin\varphi}\, r\,d\varphi = -\int_{\varphi_1}^{\varphi_2}\frac{1}{r}\, r\,d\varphi = \varphi_2-\varphi_1$$

where the angles φ_2, φ_1 correspond to the endpoints of a_i.

Now an elementary geometric argument (IMPLICIT EXERCISE) yields the result. □

WLOG, we may always apply a hyperbolic isometry (fixing two pts in ∂H) and assume that one side of the triangle is the imaginary axis. Thus, we may regard every hyperbolic triangle whose vertices are not on the same geodesic as



Cor. 2.5.1 Hyperbolic area of the hyperbolic triangle w/ interior angles $\alpha_1, \alpha_2, \alpha_3$ is given by

$$\text{Area}(B) = \pi - \sum_{i=1}^{3}\alpha_i. \qquad \square$$

The result of Thm 2.5.1 is actually valid for "quite general metrics", not just the hyperbolic metric. In fact, the Euclidean case is trivial, and the ~~corollary~~ Theorem can be proved for the spherical metric just as above.

We generalize,

Corollary 2.5.2. Let P be a geodesic polygon in H w/ k vertices, of interior angles $\alpha_1, \dots, \alpha_k$. Then

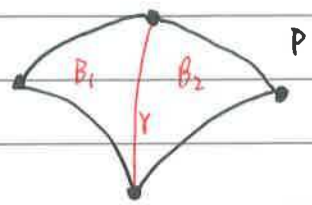
$$\int_P K_H \lambda_H^2 \frac{i}{2}\, dz\wedge d\bar z = \int_P \frac{1}{y^2}\frac{1}{2i}\, dz\wedge d\bar z = \sum_{i=1}^{k}\alpha_i + (2-k)\pi$$

and,

$$\mathrm{Area}_{H}(P) = (k-2)\pi - \sum_{i=1}^{k} \alpha_i .$$

Idea of Pf. Decompose the polygon into triangles

e.g.,



γ a geodesic contained in P.

If P has k vertices, we can decompose it into $(k-2)$ geodesic triangles. The result follows. □

Corollary 2.5.3. Suppose $\Gamma < PSL(2,\mathbb{R})$ acts freely and properly discontinuously on H and $\Gamma\backslash H$ is compact. Then

$$\int_{\Gamma\backslash H} K \frac{1}{y^2} \frac{i}{2}\, dz \wedge d\bar{z} = 2\pi(2-2p)$$

and

$$\mathrm{Area}_{H}(\Gamma\backslash H) = 2\pi(2p-2)$$

where $4p$ is the number of sides of the (Dirichlet) fundamental polygon for Γ in H. (via Thm. 2.4.2.)

Proof. Follows directly from Thm. 2.4.2 and Cor 2.5.2. □

Def'n 2.5.1 The p of this corollary is the genus of the of the compact Riemann surface $\Gamma\backslash H$, and $\chi = 2-2p$ is the Euler characteristic.

Rmk. Corollary 2.5.3's result may be restated as

$$\int_S K_\lambda \, dA_\lambda = 2\pi\chi(S) \qquad \text{for } S = \Gamma\backslash H.$$

Cor 2.5.4 (of Thm 2.4.2)

For a Riemann surface $S = \Gamma \backslash H$, $p > 1$ and $\chi(S) < 0$.

Rmks. As a consequence of the Uniformization Thm (Th 4.4.1), in fact, every compact Riemann surface is conformally equivalent to S^2, a torus $\mathbb{C}/M$ (M a module over $\mathbb{Z}$ of rank 2), or a surface $\Gamma \backslash H$.

since the universal covering must be S^2, $\mathbb{C}$, or H.

But S^2 admits no nontrivial quotients and the compact quotients of $\mathbb{C}$ are the tori.

As we've already discussed, $p = 0$ for S^2 and $p = 1$ for T^2, so that $\chi(S^2) = 2$ and $\chi(T^2) = 0$.

Thus the genus (or equivalently, Euler characteristic) already determines the conformal type of the universal covering.

Cor 2.5.5

Given a Riemann surface S decomposed into Geodesic polygons, w/ "f" faces, "e" edges, and "v" vertices. [NOTE the change of notation from JJ!] Then

$$\chi(S) = v - e + f.$$

Proof. We stated this as our def'n, but now it follows directly from the definition of genus above and the previous corollaries. $\square$

Remark. As a purely topological object, the previous corollary holds even if the polygons are not "geodesic." (Recall, geodesics depend on the conformal metric, so they are a geometric object.)

We now state the Gauss-Bonnet Theorem (as a corollary, oddly enough) for closed Riemann surfaces w/ arbitrary conformal metric.

Cor. 2.5.6 (Gauss-Bonnet Theorem)

Let S be a compact Riemann surface without boundary (i.e., closed) of genus p, with metric $g_\rho = \rho^2(z)\,dz\,d\bar{z}$ and curvature K_ρ. Then

$$\int_S K_\rho \, dA_\rho = \int_S K_\rho \, \rho^2(z) \, \frac{i}{2} \, dz \wedge d\bar{z} = 2\pi(2-2p) = 2\pi\,\chi(S).$$

Remark. What makes this theorem remarkable is that it relates a topological invariant (the Euler characteristic, or genus) to an "arbitrary" geometric quantity. Any admissible conformal metric on S must obey this formula. Therefore there is a topological obstruction on S that permits or forbids proposed conformal metrics to/from being actual metrics. (This can be said much better, but I think you get the gist.)

Proof of Gauss-Bonnet. By the previous discussion, every compact Riemann surface admits a metric of constant curvature ($K \equiv -1, 0,$ or 1).

Let $g_\lambda = \lambda^2(z)\,dz\,d\bar{z}$ be a metric of constant curvature on S. For this metric, Cor. 2.5.3 applies and yields

$$\int_S K_\lambda \lambda^2 \tfrac{i}{2} dz \wedge d\bar{z} = 2\pi \chi(S).$$

Now, the quotient $\frac{\rho^2(z)}{\lambda^2(z)}$ is invariant under coordinate transformations since by Def'n 2.3.1 they both get multiplied by the same factor.

We compute,

$$\int_S K_\lambda \lambda^2 \tfrac{i}{2} dz \wedge d\bar{z} - \int_S K_\rho \rho^2 \tfrac{i}{2} dz \wedge d\bar{z} = 4\left(-\int_S \frac{\partial^2}{\partial z \partial \bar{z}} \log(\lambda) \tfrac{i}{2} dz \wedge d\bar{z} + \int_S \frac{\partial^2}{\partial z \partial \bar{z}} \log \rho \tfrac{i}{2} dz \wedge d\bar{z}\right)$$

$$= 4 \int_S \frac{\partial^2}{\partial z \partial \bar{z}} \log\left(\frac{\rho}{\lambda}\right) \tfrac{i}{2} dz \wedge d\bar{z}$$

which vanishes by Green's Thm (or the divergence theorem) and we say in HW 2.3. Thus $\int_S K_\rho dA_\rho = \int_S K_\lambda dA_\lambda = 2\pi \chi(S)$. □

<u>Remark</u>. The Chern classes and Atiyah-Singer Index theorem are other examples of objects/theorems relating topological invariants to "non-topological" quantities.
There are many others. These fall under the umbrella of topological obstruction theory.

In fact, the curvature K_ρ defines the first Chern-class of the surface S,

$$K_\rho \rho^2 dz \wedge d\bar{z} = -4 \frac{\partial^2}{\partial z \partial \bar{z}} \log(\rho) dz \wedge d\bar{z}$$

up to a factor. It does not depend on the choice of conformal metric ρ.

Now consider a non-constant holomorphic map $f: S_1 \to S_2$ of compact Riemann surfaces.
Choose for each $p \in S_1$ local charts $U_1 \ni p$ and $V_1 \ni f(p)$ such that $p = 0 = f(p)$ and f can be written as

$$f(z) = z^n. \qquad (*)$$

Defn 2.5.2 p is called a branch point or ramification point of f iff $n > 1$ in $(*)$. We call $n-1$ the order of ramification of f at p, and write $N_f(p) := n-1$.

Note: S_1 compact $\Rightarrow \exists$ only finitely many ramification pts.

Lemma 2.5.1
Let $f: S_1 \to S_2$ be a non-constant holomorphic map of compact Riemann surfaces. Then there exists $m \in \mathbb{N}$ s.t.

$$\sum_{p \in f^{-1}(q)} (N_f(p) + 1) = m$$

for all $q \in S_2$. Thus f takes every value in S_2 precisely m times, multiplicities taken into account.

Defn 2.5.3 We call m the (mapping) degree of f. If f is constant, we set $m = 0$.

* The proof of Lemma 2.5.1 follows from a "simple" open-closed argument. (implicit exercise!) □

We now are ready to state

<u>Thm 2.5.2</u> <u>(Riemann-Hurwitz)</u>
Let $f: S_1 \to S_2$ be a non-constant holomorphic map of degree m between compact Riemann surfaces of genera g_1 and g_2, respectively. Let $\nu_f := \sum_{p \in S_1} \nu_f(p)$ be the total order of ramification of f. Then

$$2-2g_1 = m(2-2g_2) - \nu_f .$$

<u>Remark</u>. Again, the result is more elegantly stated in terms of the Euler characteristic,

$$\chi(S_1) = m\,\chi(S_2) - \nu_f .$$

<u>Proof</u>. Let $g_\lambda = \lambda^2(w)\,dw\,d\bar{w}$ be a conformal metric on S_2. Then

$$(f^* g_\lambda) = \lambda^2(f(z)) \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}}\, dz\,d\bar{z}$$

defines a conformal metric on S_1 outside of the ramification pts of f, and f is a local isometry wrt these two metrics.

Let $p_1, \dots, p_k$ be the ramification pts of f. Suppose f is given in a local chart near p_j by $w = f(z) = z^{\nu_f}$, and let $B_j(r)$ be a disk of radius r around p_j in this chart.

Since f is a local isometry, as $r \to 0$ we have

$$-\frac{1}{2\pi}\int_{S_1 \setminus \bigcup_{j=1}^k B_j(r)} 4\frac{\partial^2}{\partial z \partial \bar{z}} \log\left(\lambda\left(\frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}}(z)\right)^{\frac{1}{2}}\right) \frac{i}{2}\, dz \wedge d\bar{z}$$

$$= \frac{-m}{2\pi}\int_{S_1 \setminus \bigcup_{j=1}^k B_j(r)} 4\frac{\partial^2}{\partial w \partial \bar{w}}(\log \lambda)\, \frac{i}{2}\, dw \wedge d\bar{w} \to m(2-2g_2) = \chi(S_2)\cdot m \quad \text{by Cor. 2.5.6.}$$

On the other hand, as $r \to 0$

$$-\frac{1}{2\pi}\int_{S_1\setminus\bigcup_{j=1}^{k} B_j(r)} 4\frac{\partial^2}{\partial z\partial\bar{z}}(\log\lambda)\,\frac{i}{2}\,dz\wedge d\bar{z} \to \chi(S_1) = 2-2g_1 .$$

Moreover, we have

$$-\frac{1}{2\pi}\int_{S_1\setminus\bigcup_{j=1}^{k} B_j(r)} 4\frac{\partial^2}{\partial z\partial\bar{z}}\left(\log\left(\frac{\partial w}{\partial z}\right)^{1/2}\right)\frac{i}{2}\,dz\wedge d\bar{z} = \frac{1}{2\pi}\sum_{j=1}^{k}\int_{\partial B_j(r)} \frac{\partial}{\partial r}\log\left(\frac{\partial w}{\partial z}\right)^{1/2} r\,d\varphi$$

$$= \frac{1}{2}\sum_{j=1}^{k}(n_j-1)$$

since $w = z^{n_j}$ in $B_j(r)$,

and similarly for $\left(\frac{\partial\bar{w}}{\partial\bar{z}}\right)^{\frac{1}{2}}$. Taken together, these imply the result. □

Some consequences of Riemann-Hurwitz:

Corollary 2.5.7

1.) N_f is always even
2.) $g_1 \geq g_2$ (so $\chi(S_1) \leq \chi(S_2)$)
3.) $g_2 = 0$, f unramified $\Rightarrow$ $g_1 = 0$, $m = 1$.
4.) $g_2 = 1$, f unramified $\Rightarrow$ $g_1 = 1$, m arbitrary.
5.) $g_2 > 1$, f unramified $\Rightarrow$ $g_1 = g_2$, $m = 1$, or $g_1 > g_2$, $m > 1$.
6.) $g_2 = 1 = g_1$ $\Rightarrow$ f unramified
7.) $g_2 = g_1 > 1$ $\Rightarrow$ $m = 1$, f unramified. □