Riemann Surfaces
Lecture 2

At the end of last class we were talking about examples of smooth manifolds. Here's another example of a compact 2-dimensional manifold.

Example The flat torus, $\mathbb{T}^2$.

Homotopy and the Fundamental Group

The material in this section does not require smoothness, but we may as well assume it as everything we study in this course will be smooth.

Definition Homotopy of maps.

Example Any two paths $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^n$ with the same endpoints are homotopic.

Example Any two paths $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^n$ which are reparametrizations of each other are homotopic.

Definition Concatenation.

Definition The fundamental group.

Theorem *$\pi_1(M, p_0)$ is a group with respect to the operation of multiplication of homotopy classes (concatenation). The identity is the class of the constant path $\gamma_0 \equiv p_0$.*

Proof

Lemma *For any $p_0, p_1 \in M$, the groups $\pi_1(M, p_0)$ and $\pi_1(M, p_1)$ are isomorphic.*

Proof

Definition The fundamental group (again).

Remark

Definition Simply connected.

Lemma *If M is simply connected, then any two paths $\gamma_1, \gamma_2 : [0,1] \to M$ with*

$$\gamma_1(0) = \gamma_2(0) \text{ and } \gamma_1(1) = \gamma_2(1)$$

are homotopic.

Proof Recommended Exercise.

Example S^n is simply connected for $n \geq 2$.

Definition Null-homotopic.

Lemma *Let $f : M \to N$ be a continuous map of manifolds, and $q_0 := f(p_0)$. Then f induces a morphism*

$$f_* : \pi_1(M, p_0) \to \pi_1(N, q_0)$$

of fundamental groups.

Proof