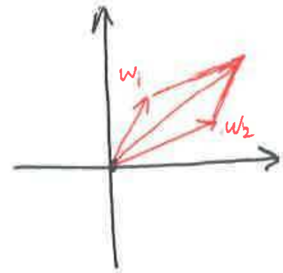
First: Atlas, differential structure

RS as complex curves, real surfaces, nat. setting for complex functions.

Riemann Surfaces
Lecture 2

At the end of last class we were talking about examples of smooth manifolds. Here's another example of a compact 2-dimensional manifold.

$\mathbb{C}\setminus\{0\}$



Example The flat torus, $\mathbb{T}^2$.

Let $w_1, w_2 \in \mathbb{C}_0$ s.t. $\frac{w_1}{w_2} \notin \mathbb{R}$.

Say $z_1 \sim z_2$ iff $z_2 - z_1 = m w_1 + n w_2$ for some $m, n \in \mathbb{Z}$.

$t w_2 \sim t w_2 + w_1$  cylinder

$s w_1 \sim s w_1 + w_2$  flat torus

Charts: $(\Delta_\alpha, (\pi|\Delta_\alpha)^{-1})$ where Δ_α contains no two equivalent points, and π is the projection mapping pts to their equiv. class.

Homotopy and the Fundamental Group

The material in this section does not require smoothness, but we may as well assume it as everything we study in this course will be smooth.

Definition Homotopy of maps.

Two continuous maps $f_1, f_2 : M \to N$ of manifolds are homotopic iff $\exists$ a continuous map $F: M \times [0,1] \to N$ with $F|_{M\times\{0\}} = f_1$ and $F|_{M\times\{1\}} = f_2$. We write $f_1 \approx f_2$.

Let $\gamma_1, \gamma_2 : [0,1] \to M$ be paths w/ $\gamma_1(0) = \gamma_2(0) = p_0$ and $\gamma_1(1) = \gamma_2(1) = p_1$. We say γ_1 and γ_2 are homotopic iff $\exists$ a continuous map $G: [0,1]\times[0,1] \to M$ s.t.

$$\begin{cases} G|_{\{0\}\times[0,1]} = p_0 & G|_{\{1\}\times[0,1]} = p_1 \\ G|_{[0,1]\times\{0\}} = \gamma_1 & G|_{[0,1]\times\{1\}} = \gamma_2 \end{cases}$$

We write $\gamma_1 \approx \gamma_2$.

*- Homotopies must keep endpoints fixed.

Example Any two paths $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^n$ with the same endpoints are homotopic.

The homotopy is $G(t,s) := (1-s)\gamma_1(t) + s\gamma_2(t)$.

Example Any two paths $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^n$ which are reparametrizations of each other are homotopic.

Fact. Each homotopy class contains a smooth representative. [PBR1,].

If $\tau : [0,1] \to [0,1]$ is continuous and strictly increasing $\gamma_2(t) = \gamma_1(\tau(t))$, we can put $G(t,s) := \gamma_1((1-s)t + s\tau(t))$.

The homotopy class of a path is its equivalence class under homotopy equiv. (AE). It is independent of parametrization.

Definition Concatenation.

Let $\gamma_1, \gamma_2 : [0,1] \to M$ w/ $\gamma_1(1) = \gamma_2(0)$. Then $\gamma_2\gamma_1 := \gamma$ is given by

$$\gamma := \begin{cases} \gamma_1(2t) & t \in [0, 1/2] \\ \gamma_2(2t-1) & t \in [1/2, 1]. \end{cases}$$

If $\gamma_1 \approx \bar{\gamma}_1$ and $\gamma_2 \approx \bar{\gamma}_2$, then $\gamma_2\gamma_1 \approx \bar{\gamma}_2\bar{\gamma}_1$. Thus this operation depends only on homotopy class — therefore we can define multiplication on homotopy classes.

Definition The fundamental group.

For any $p_0 \in M$, the space $\pi_1(M, p_0)$ is the group of homotopy classes of paths $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1) = p_0$; i.e., closed paths $\gamma : S^1 \to M$ w/ $\gamma(1) = p_0$.

$\pi_1(M, p_0)$ is called the fundamental group of M at p_0.

Theorem *$\pi_1(M, p_0)$ is a group with respect to the operation of multiplication of homotopy classes (concatenation). The identity is the class of the constant path $\gamma_0 \equiv p_0$.*

Proof

The inverse of a class $[\gamma]$ is the class of $\overleftarrow{\gamma}$, where $\overleftarrow{\gamma}(t) = \gamma(1-t)$.

Indeed, $$G(t,s) = \begin{cases} \gamma(2st), & t \in [0, \tfrac{1}{2}] \\ \overleftarrow{\gamma}(1+2s(t-1)) = \gamma(2s(1-t)), & t \in [\tfrac{1}{2}, 1] \end{cases}$$

is a homotopy between $\gamma_0 = p_0$ and $\overleftarrow{\gamma} \circ \gamma$. (RE)

Lemma *For any $p_0, p_1 \in M$, the groups $\pi_1(M, p_0)$ and $\pi_1(M, p_1)$ are isomorphic.*

Proof

Choose a curve γ w/ $\gamma(0) = p_0$ and $\gamma(1) = p_1$. If g is a path w/ $g(0) = g(1) = p_1$, then $\mathcal{H}_\gamma(g) = \gamma^{-1} \circ g \circ \gamma$ maps $\pi_1(M, p_1)$ to $\pi_1(M, p_0)$. This map (called conjugation) is an isomorphism (RE).

Definition The fundamental group (again).

If (since) M is (path) connected, then the lemma implies there is a unique fundamental group $\pi_1(M)$ of M.

Remark

The isomorphism of the lemma depends on the choice of γ, e.g., suppose $\gamma(0) = \gamma(1) = p_0$ (so $p_0 = p_1$). Then $\mathcal{H}_\gamma$ is in general, a non-trivial automorphism of $\pi_1(M, p_0)$.

Definition Simply connected.

M is said to be simply connected iff $\pi_1(M) = \{1\}$.

Lemma *If M is simply connected, then any two paths $\gamma_1, \gamma_2 : [0,1] \to M$ with*

$$\gamma_1(0) = \gamma_2(0) \text{ and } \gamma_1(1) = \gamma_2(1)$$

are homotopic.

Proof Recommended Exercise.

Example S^n is simply connected for $n \geq 2$.

Definition Null-homotopic.

Let $\gamma_0 \equiv p_0$ be the constant path. A path $\gamma_1 : [0,1] \to M$ w/ $\gamma_1(0) = \gamma_1(1) = p_0$ is said to be null-homotopic iff $\gamma_1 \approx \gamma_0$.

Lemma *Let $f : M \to N$ be a continuous map of manifolds, and $q_0 := f(p_0)$. Then f induces a morphism*

$$f_* : \pi_1(M, p_0) \to \pi_1(N, q_0)$$

of fundamental groups.

Proof

If $\gamma_1 \approx \gamma_2$, then $f(\gamma_1) \approx f(\gamma_2)$ by continuity. then $f(\overleftarrow{\gamma_2} \cdot \gamma_1) \approx \overleftarrow{f(\gamma_2)} \cdot f(\gamma_1)$ and f_* is a well-defined map of fundamental groups, preserving the group ops.