**Defn 2.4.6** A fundamental polygon $F$ of the type defined in Thm 2.4.1 is called the metric or Dirichlet fundamental polygon at $z_0$.

**Sketch of Proof of Thm 2.4.1.**

Since $\Gamma \backslash H$ is compact, $F$ is bounded.

For each $g \in \Gamma$, the curve $l := \{z \in H \mid d(z, z_0) = d(z, gz_0)\}$ is a geodesic.

Since $F$ is bdd, $\bar{F}$ is cpt, and there can exist only finitely many $g \in \Gamma$ s.t.

$$d(z, z_0) = d(z, gz_0) \quad \text{for some } z \in \bar{F}.$$

In fact $F$ is the intersection of finitely many "half planes" $\{z \in H \mid d(z, z_0) < d(z, gz_0)\}$.
Let $g_1, \dots, g_m$ be the elements from the previous argument.

In particular, $F$ is convex and has finitely many edges, all of which are geodesic arcs.
The intersection of any two arcs, when non-empty, is a common vertex, and interior angles are less than $\pi$.

To show that every $z \in H$ has an equivalent pt in $\bar{F}$, first determine a $g \in \Gamma$ s.t.

$$d(z, gz_0) \le d(z, g' z_0) \quad \forall g' \in \Gamma.$$

Then $g^{-1} z$ is equivalent to $z$ and it lies in $\bar{F}$ since the action of $\Gamma$ is by isometries (preserves distance).

Conversely, if $z \in F$, then

$$d(z, z_0) < d(z, gz_0) = d(g^{-1}z, z_0)$$

for all $g \in \Gamma$, so there is no other pt equivalent to $z$ in $F$.
Thus $F$ is a fundamental polygon.

An edge $\sigma_i$ of $F$ is given by

$$\sigma_i = \{ z \in H \mid d(z, z_0) = d(z, g_i z_0),\ d(z, z_0) \leq d(z, g z_0) \text{ for all } g \in \Gamma \}$$

Since $d(g_i^{-1} z, g z_0) = d(z, g_i g z_0)$, we have for $z \in \sigma_i$

$$d(g_i^{-1} z, z_0) \leq d(g_i^{-1} z, g z_0) \qquad \text{for all } g \in \Gamma$$

w/ equality precisely when $g = g_i^{-1}$. Thus, $g_i^{-1}$ carries $\sigma_i$ to another edge.
Since $F$ is a convex polygon w/ interior angles $< \pi$, $g_i^{-1}$ carries all other edges outside of $F$.
Thus different pairs of edges are carried to each other by different transformations.
Further, the transformation that carries $\sigma_i$ to another side is uniquely determined since, for any interior pt of $\sigma_i$, equality holds in the previous inequality only for $g = g_i$.

Thus, all assertions of the theorem are proved. $\square$

**Cor 2.4.1** The transformations $g_1, \dots, g_m$ of the proof generate $\Gamma$.

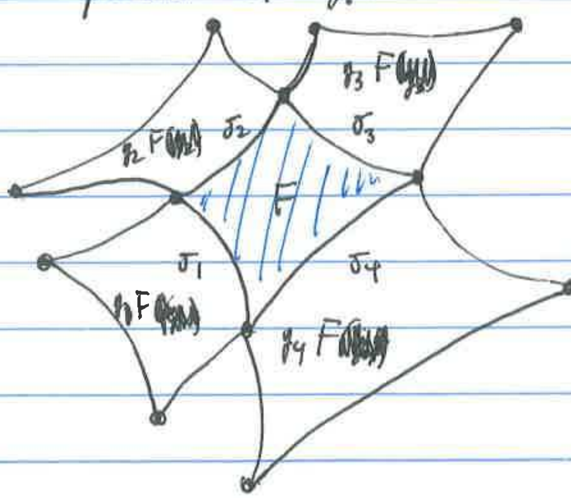
**Proof.** ~~For any $g \in \Gamma$, consider the Dirichlet polygon wrt $gz_0$, $F(g)$. Among the $F(g')$, only the $F(g_i)$ have a side in common w/ $F(g)$, and $g_i^{-1}$ carries $F(g_i)$ to $F(g)$~~

For $g \in \Gamma$, let $F(g)$ denote the Dirichlet polygon wrt $gz_0$.
$F$ still denotes the Dirichlet polygon at $z_0$.
Among the $F(g')$, only the $F(g_i)$ have a side in common w/ $F$, and $g_i^{-1}$ carries $F(g_i)$ to $F$.

If now $F(g')$ has a side in common w/ $F(g_i)$ say, then $g_i^{-1}F(g')$ has a side in common w/ $F$ so that there exists $j \in [1, \dots, m]$ w/ $g_j^{-1} g_i^{-1} F(g') = F$.

Now any $F(g_0)$ can be joined to $F$ by a chain of the $F(g)$ for which each two successive elements have a side in common.

Hence $F(g_0)$ can be carried to $F$ by a product of $g_i^{-1}$, and $g_0$ is a product of $g_i$. $\square$



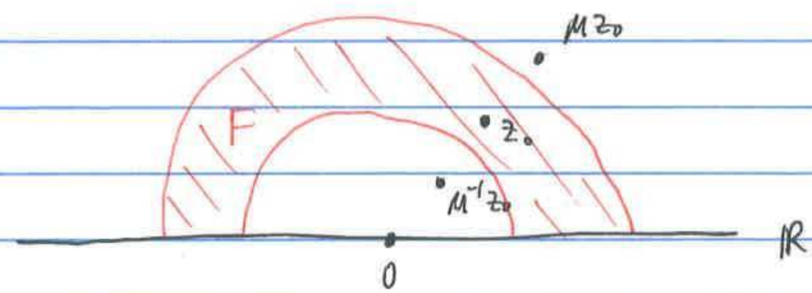

looks "something" like this

Rmk. $\mathbb{H} = \bigcup_{g \in \Gamma} g\bar{F}$.

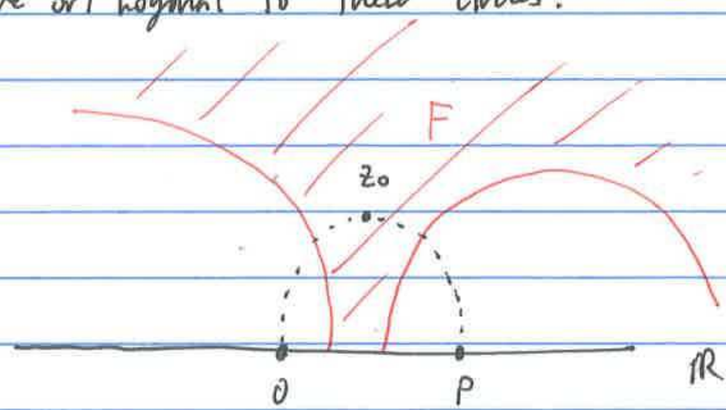
Example. Suppose $\Gamma$ is cyclic (has one generator) and this generator $g_1$ is a hyperbolic automorphism of $\mathbb{H}$. The two fixed pts in $\mathbb{R} \cup \{\infty\}$ are WLOG $0$ and $\infty$.

Then $g_1 z = \mu z$, $\mu > 0$.

Thus the pts equivalent to $z_0$ lie on the ray from the origin through $z_0$, and $F$ will be bounded by two circles orthogonal to these rays and the x-axis.



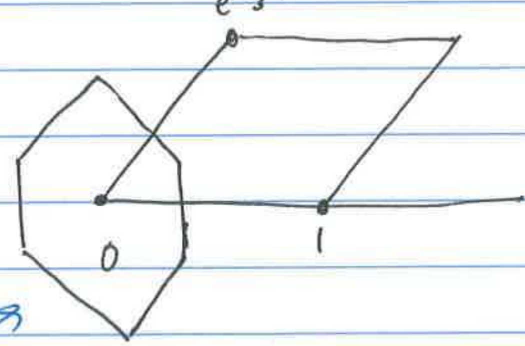

If we were to take a generator $g_1$ w/ fixed pts $0$ and $P \in \mathbb{R}$, the $g z_0$ lie on a circle through $0, z_0, P$, and the sides of $F$ are orthogonal to these circles.



Example. Finally, consider the groups $\Gamma$ of Euclidean motions. In the compact cases, $\mathbb{C}/\Gamma$ (or $\Gamma\backslash\mathbb{C}$) is a torus.

In this case, F is a fundamental hexagon.

e.g.) if $\Gamma = \{z \mapsto z+n+me^{\frac{2\pi i}{3}},\ n,m\in\mathbb{Z}\}$ then one obtains a regular hexagon.




pretend this is "regular".

Thm 2.4.2 Under the assumptions of Thm 2.4.1, there exists a fundamental polygon w/ finitely many sides, all of whose vertices are equivalent. Here again, every $a$ is carried by precisely one element of $\Gamma$ to another side $a'$, and the transformations corresponding in this way to distinct pairs of equivalent sides are distinct. The sides will be described in the order

$$a_1 b_1 a_1' b_1' a_2 b_2 \cdots a_p b_p a_p' b_p' ;$$

in particular, the number of sides is divisible by 4.

* This proof is extremely technical and we'll probably skip it so that we can keep moving forward.

Now we mention some results concerning the fundamental group of a surface $\Gamma\backslash\mathbb{H}$.

Thm 2.4.3. Let $\Gamma\backslash\mathbb{H}$ be a compact Riemann surface of genus $p>1$. Then the fundamental group $\pi_1(\Gamma\backslash\mathbb{H}, p_0)$ has $2p$ generators $a_1, b_1, a_2, \ldots, a_p, b_p$, w/ the single defining relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_p b_p a_p^{-1} b_p^{-1} = 1.$$

and

Cor 2.4.2. Every compact Riemann surface of the form $\Gamma\backslash\mathbb{H}$ has a non-abelian fundamental group. □

Recall that the Euler characteristic of a surface is

$$\chi = v - e + f$$

for any triangulation of the surface.

The genus of a closed surface is given implicitly by

$$\chi = 2 - 2g,$$

The genus of a compact Riemann surface w/out boundary (closed) is then the number of handles.

This number (or equivalently the Euler characteristic) completely characterizes compact Riemann surfaces. More generally,

orientability.

Cor 2.4.A.2 Two smooth, orientable, compact, triangulated surfaces are homeomorphic iff they have the same genus.

Next week - Gauss-Bonnet, Riemann-Hurwitz, and Schwarz!