Def'n 2.3.7 Let G be a group and M a manifold. M is said to be a (left) G-manifold if there is an action $\tau: G \times M \to M$ of G on M w/

$$\tau_{g_1}(\tau_{g_2} x) = \tau_{g_1 g_2}(x) \qquad \forall g_1, g_2 \in G, \ x \in M$$

and $\tau_e x = x$ for all $x \in M$. (Better - just use $e = 1$)

(e is the identity of G.)

Then such a group (together w/ the action) is called a transformation group of M.

We are interested in the case when G is a (sub) group of isometries of M. (hence, M must have a Riemannian metric.)

RE The isometries of a Riemannian manifold (M, g) constitute a group of transformations.

Def'n. Let M be a G-manifold and $p \in M$. The isotropy group of $p \in M$ is

$$H_p = \{ g \in G \mid \tau_g(p) = p \}$$

i.e., all of the elements of G that fix p in M.

Def'n. A group action $\tau: G \times M \to M$ is said to be transitive if and only if for ~~and~~ any $p_1, p_2 \in M$, there exists a $g \in G$ w/ $\tau_g p_1 = p_2$.

Defn. An action $\tau : G \times M \to M$ is said to be effective (or faithful) if and only if $\tau_g(p) = p$ implies $g = 1$.

Defn.
$GL(2,\mathbb{R}) = \{ A \in \mathbb{R}^{2\times 2} \mid \det(A) \neq 0 \}$
$SL(2,\mathbb{R}) = \{ A \in GL(2,\mathbb{R}) \mid \det(A) = 1 \}$
$PSL(2,\mathbb{R}) = SL(2,\mathbb{R}) / \{\pm Id\}$
$O(2) = \{ Q \in GL(2,\mathbb{R}) \text{ w/ } Q^T Q = I \}$
$SO(2) = \{ Q \in O(2) \mid \det Q = 1 \}$

Thm 2.3.4 $PSL(2,\mathbb{R})$ is a transformation group of H. The action is transitive and effective. For any $p \in H$, the isotropy group is $H_p \cong SO(2)$.

Proof. RE-Read through the details. $\square$

Lemma 2.3.5 Actually, $PSL(2,\mathbb{R}) = Iso(H)$ wrt the hyperbolic metric
$g_H = \frac{1}{y^2} dz d\bar{z}$. $\square$

Defn 2.4.1 An action of G on M is said to be properly discontinuous iff every $p \in M$ has a nbd such that $\{ g \in G \mid \tau_g(U) \cap U \neq \emptyset \}$ is finite, and if p_1, p_2 are not in the same orbit, ~~[illegible]~~, ~~[illegible]~~ then they have nbds U_1, U_2 with $\tau_g(U_1) \cap U_2 = \emptyset$ for all $g \in G$. (see below)

The orbit of a point $p \in M$ wrt a left action is the set
$\{ q \in M \mid \exists\, g \in G \text{ w/ } \tau_g(p) = q \}$ or equivalently
$\{ \tau_g(p) \in M \mid g \in G \}$

Lemma 2.4.1 If G acts properly discontinuously on M, then the orbit of every $p \in M$ is discrete (has no accumulation pt). $\blacksquare$

We now wish to study properly discontinuous subgroups Γ of $PSL(2,\mathbb{R})$ acting on $\mathbb{H}$. (by isometries).
The assumption that Γ acts properly discontinuously forces Γ to be a discrete subgroup of $PSL(2,\mathbb{R})$ (i.e., a lattice).
Indeed, if $(g_n) \to g$ in Γ, then for every $p \in \mathbb{H}$ $(T_{g_n}(p)) \to T_g p \in \mathbb{H}$.
In particular Γ is countable.

Def'n 2.4.2 ~~[illegible]~~
Let G be a transformation group of M and Γ a lattice in G.
Two points $p_1, p_2 \in M$ are said to be equivalent wrt the action of Γ iff $\exists$ $g \in \Gamma$ w/ $T_g p_1 = p_2$.
Let M/Γ, ~~[illegible]~~ or better yet $\Gamma \backslash M$ since the action is on the left, be the space of equivalence classes equipped w/ the quotient topology.

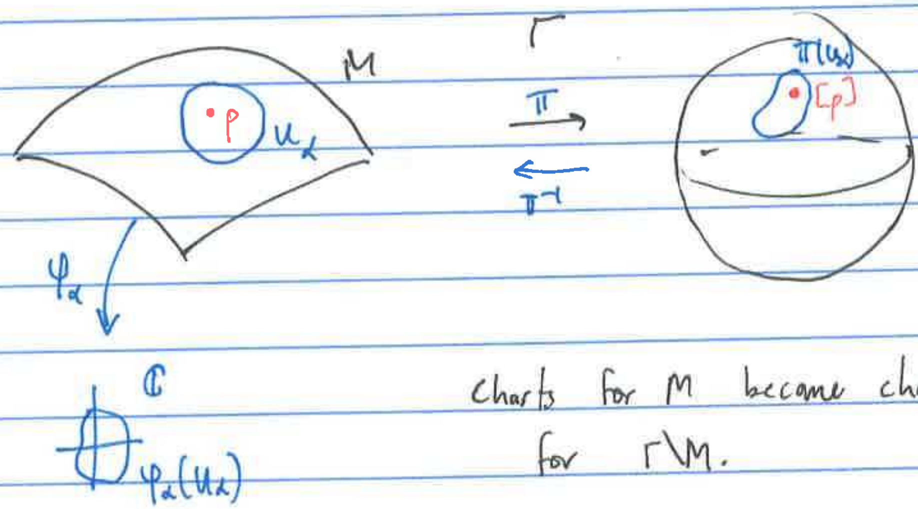
Def'n An action of G on M is free (of fixed points) iff $T_g(p) \neq p$ for all $p \in M$ and all $g \neq 1 \in G$.

If the action of Γ is free, and M is a Riemann surface w/ conformal metric, then $\Gamma \backslash M$ $(= M/\Gamma)$ is a Riemann surface and inherits a conformal metric.

For $[p] \in \Gamma \backslash M$, choose $q \in \pi^{-1}([p])$. Since Γ acts freely and properly discontinuously, q has a nbd U such that $T_g(U) \cap U = \emptyset$

for $g \neq 1 \in \Gamma$, so that $\pi : U \to \pi(U) \in \Gamma \backslash M$ is a homeomorphism. The conformal metric is carried to the quotient since Γ acts by isometries.

Henceforth, we let $M = H$, and Γ be a lattice of $PSL(2, \mathbb{R})$. Then $\Gamma \backslash H$ is a hyperbolic Riemann surface.



Charts for M become charts for $\Gamma \backslash M$.

Now, recall $SL(2, \mathbb{R}) = \{ A \in \mathbb{R}^{2\times 2} \mid \det(A) = 1 \}$. Then $SL(2,\mathbb{R})$ acts on H via

$$z \longmapsto \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$$

Lemma 2.4.2 Each $\gamma \in SL(2,\mathbb{R})$, $\gamma \neq I$, ~~has~~ either has one fixed point in H, one fixed point on the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} = \partial H$, or two fixed points on ∂H.

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, this corresponds to $|\operatorname{tr} \gamma| > 2$, $|\operatorname{tr} \gamma| = 2$, or $|\operatorname{tr} \gamma| < 2$, resp.

Recall $\operatorname{tr} \gamma = a + d$.

Proof. If z is a fixed point of γ, then
$$cz^2 + (d-a)z - b = 0.$$
Completing the square,
$$z = \frac{a-d}{2c} \pm \frac{\sqrt{(a-d)^2 + 4bc}}{2c}$$
$$= \frac{1}{2c}\left[(a-d) \pm \sqrt{(a+d)^2 - 4}\right].$$
(since $ad - bc = 1$)
The conclusion follows easily. $\square$

Def'n 2.4.3 An element of $SL(2,\mathbb{R})$ w/ one fixed pt in H is called elliptic, an element w/ one fixed pt on $\bar{\mathbb{R}}$ parabolic, and one w/ two fixed pts on $\bar{\mathbb{R}}$ hyperbolic.

Example. Elliptic automorphisms of H may be regarded as fixing i,
$$\frac{ai+b}{ci+d} = i.$$
One example is $z \mapsto \frac{-1}{z}$. This also maps 0 to ∞, and maps geodesics through i to themselves, but w/ direction reversed.

Elliptic automorphisms of the disk D (which is conformally equivalent to H) are maps that fix $0 \in D$ — rotations $z \mapsto e^{i\alpha z}$, $\alpha \in \mathbb{R}$.

Example Let $\mu \in \mathbb{R}$, $\mu \neq 1$. A transformation $z \mapsto \mu^2 z$ may be regarded as $\gamma_\mu = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \in SL(2,\mathbb{R})$. This map fixes 0 and ∞, hence is hyperbolic.

All such hyperbolic automorphisms may be regarded as such (via Möbius transformations of $\mathbb{H}$).

This γ_μ leaves the geodesic connecting 0 and ∞ (the imaginary axis) invariant.

The pts on the imaginary axis are shifted (translated) by the distance

$$\int_{\bar{y}i}^{\mu^2 \bar{y} i} \frac{1}{y}\,dy = \log \mu^2 .$$

Example. The transformation $\gamma : z \mapsto z+1$ regarded as $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{R})$ has a single fixed pt, $\infty \in \overline{\mathbb{R}}$, hence is parabolic. Again any such may be regarded as $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{R})$.

Similarly $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} : z \mapsto \frac{1}{cz+1}$ has a single fixed pt, $0 \in \overline{\mathbb{R}}$, hence is also parabolic.

Lemma 2.4.3. Let $\Gamma \backslash \mathbb{H}$ be a compact Riemann surface for a lattice Γ of $PSL(2,\mathbb{R})$. Then all elements of Γ are hyperbolic.

Γ must act freely and properly discontinuously!

Proof. Γ cannot be elliptic because it cannot have fixed points in $\mathbb{H}$.

Let $\gamma \in \Gamma$. Since $\Gamma \backslash \mathbb{H}$ is compact, $\exists\, [z_0] \in \Gamma \backslash \mathbb{H}$ w/

$$d(z_0, \gamma z_0) \leq d(z, \gamma z) \quad \text{for all } z \in \mathbb{H}. \qquad (*)$$

where d denotes hyperbolic distance.

Now assume γ is parabolic, WLOG of the form $\gamma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.

then for each $z \in H$, $d(z, \gamma z) = d(z, z+b)$, and this tends to 0 and $\operatorname{Im} z \to \infty$. Thus a parabolic γ cannot satisfy the previous inequality, $(*)$ as γ has no fixed pt in H. $\square$

<u>Lemma 2.4.4</u> Let $\Gamma \backslash H$ be a compact Riemann surface. Then for each $\gamma \in \Gamma$, $\gamma \neq I$, the free homotopy class of loops determined by γ contains precisely one closed geodesic.

<u>Proof.</u> WLOG $\gamma \in \Gamma$ has 0 and ∞ as fixed pts, so $\gamma = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$, $\mu \neq 1$, and $\exists!$ invariant geodesic of γ. The closed geodesics of $\Gamma \backslash H$ are projections of the geodesics of H that are invariant under some nontrivial $\gamma \in \Gamma$. The element $\gamma \in \Gamma$ determines the homotopy class of the projected geodesic. $\square$

<u>Rmk.</u> The length of the closed geodesic is $\log \mu^2$ in $\Gamma \backslash H$ as γ identifies pts on the imaginary axis that a this far apart.

<u>Defn 2.4.4</u> An open subset F of H is a <u>fundamental domain</u> for Γ if every $z \in H$ is equivalent under Γ to a pt z' in the closure of F, whereas no two pts of F are equivalent.

<u>Defn 2.4.5</u> A fundamental domain is a <u>fundamental polygon</u> iff ∂F is a countable union of geodesic arcs (and their limit pts), the intersection two such arcs being a vertex iff non-empty.

Continue w/ Γ free, and $\Gamma \backslash H$ compact. We now construct fundamental polygons.

Thm 2.4.1

Spse $\Gamma \subset PSL(2,\mathbb{R})$ acts properly discontinuously and w/out fixed pts on H, and that $\Gamma \backslash H$ is compact. Let $z_0 \in H$ be fixed but arbitrary. Then

$$F := \{ z \in H \mid d(z, z_0) < d(z, T_g z_0) \ \forall g \in \Gamma \}$$

is a convex fundamental polygon for Γ w/ finitely many edges. For every edge σ of F, $\exists!$ other edge σ' of F w/ $T_g \sigma = \sigma'$ for some $g \in \Gamma$. Different pairs of edges are identified by different elements of Γ.

Proof

TBD?
We at least need to sketch the outline and draw some pictures. And define a notion or two.
We'll do this tomorrow.