Some facts about the geodesic equation that hold for all Riemannian metrics (not necessarily conformal) on surfaces:

Let g be a metric tensor of the form

$$g = \sum_{j,k=1}^{2} g_{jk}(x)\, dx^j dx^k$$

The geodesic equation becomes

$$\ddot{\gamma}(t) + \sum_{j,k=1}^{2} \Gamma^i_{jk}(\gamma(t))\, \dot{\gamma}^j(t)\, \dot{\gamma}^k(t) = 0 \qquad i = 1,2$$

where

$$\Gamma^i_{jk}(x) = \frac{1}{2} \sum_{l=1}^{2} g^{il}(x) \left(\frac{\partial}{\partial x^k} g_{jl}(x) + \frac{\partial}{\partial x^j} g_{kl}(x) - \frac{\partial}{\partial x^l} g_{jk}(x) \right)$$

and (g^{il}) represents the inverse of the matrix (g_{il}).

These Γ^i_{jk} are called the Christoffel symbols of g.

We wish to use the exponential map to define geodesic polar coordinates on our surface (M,g).

Recall that for any $p \in M$, $\exp_p^{-1}$ is a (local) diffeomorphism sending p to $0 \in T_pM \cong \mathbb{R}^2$.

Let V_p denote the neighborhood of 0 in T_pM for which $\exp_p|_{V_p}$ is a diffeo, and introduce polar coordinates
$x^1 = r\cos\varphi$ and $x^2 = r\sin\varphi$.

In these coordinates, the lines $r=t$ and $\varphi=$ constant are geodesic.

We write the metric as

$$g = g_{11}\, dr^2 + 2 g_{12}\, dr\, d\varphi + g_{22}\, d\varphi^2$$

The Christoffel symbols satisfy

$$\Gamma^i_{11} = \frac{1}{2}\sum_{l=1}^{2} g^{il}\left(2\frac{\partial}{\partial r} g_{1l} - \frac{\partial}{\partial l} g_{11}\right) = 0 \qquad (\mathrm{RE})$$

Since (g^{il}) is invertible, this implies

$$2\frac{\partial}{\partial r} g_{1l} - \frac{\partial}{\partial l} g_{11} = 0 \qquad l=1,2$$

For $i=1$, we obtain $\frac{\partial}{\partial r} g_{11} = 0$.

By the properties of polar coords in general, φ is ~~[illegible]~~ undetermined for $r=0$, so

$$g_{jk}(0,\varphi)$$

is independent of φ.

This implies $\quad g_{11} \equiv$ constant.

In fact $g_{11} = 1$. (RE?)

Putting this back into

$$2\frac{\partial}{\partial r} g_{12} - \frac{\partial}{\partial 2} g_{11} = 0$$

yields $\frac{\partial}{\partial r} g_{12} = 0$.

By transformation laws of polar coordinates (from Euclidean) we have

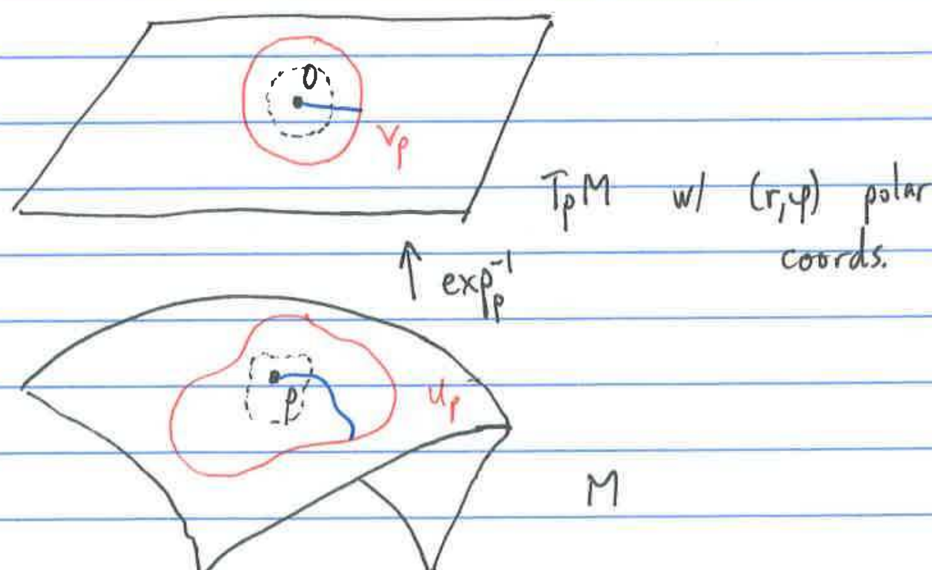
$$g_{12}(0,\varphi) = 0. \qquad \text{(RE- see p. 35)}$$

This implies $g_{12} = 0$.

Finally, we have $g_{22} > 0$ since a metric is positive definite. The metric is thus of the form

$$g = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \qquad A > 0$$

wrt to the basis $\frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi}$ of $T(U_p)$ $[U_p = \exp_p(V_p) \subset M]$.



We finish our discussion of the exponential map for now w/

Lemma 2.3.A3 Choose $\delta > 0$ s.t. $\exp_p : \{ v \in V_p \mid \|v\|_p < \delta \} \to M$ is injective. Then for every $q = \exp_p(v)$ with $\|v\|_p < \delta$, the geodesic $\gamma_{p,v}$ is the unique shortest curve from p to q. In particular,

$$d(p,q) = \|v\|_p .$$

Proof. Read it, p. 36. Makes use of the geodesic polar coords.

And,

Cor. 2.3.A.1 Let M be a compact surface w/ Riemannian metric. There exists $\varepsilon > 0$ s.t. any two points in M w/ $d(p,q) < \varepsilon$ can be connected by a unique length minimizing geodesic segment.

The geodesic segment has length $< \varepsilon$, up to reparametrization. $\square$

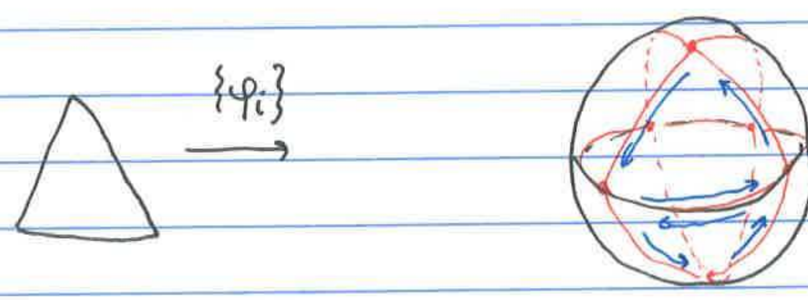
We now return to the beginning of §2.3.A and employ these results.

Let S be a compact surface (not nec. Riemann). A triangulation of S is a subdivision of S into triangles satisfying certain properties.

Def'n 2.3.A.1 A <u>triangulation</u> of a compact surface S consists of finitely many "triangles" T_i, $i=1,\dots,n$, with
$$\bigcup_{i=1}^{n} T_i = S.$$

Here a triangle is a closed subset of S homeomorphic to a plane triangle Δ. For each i we fix a homeo
$$\varphi_i : \Delta_i \to T_i$$
from a plane triangle Δ_i to T_i, and we call the images of the vertices, edges, and faces of Δ_i the same for T_i.

We require that any two triangles T_i, T_j, $i \neq j$, either be disjoint, intersect in a single vertex, or intersect in an entire edge.

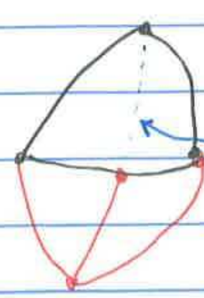


"or something"

All "directions" or orientations must cancel.

Triangulations are not unique.

Cannot have



would have to add an edge to make this acceptable.

Thm 2.3.A.1 Any compact surface endowed w/ a Riemannian metric (or admitting a Riemannian metric) can be triangulated

Remarks. This means that every compact surface — including every compact Riemann surface — can be triangulated.

Idea of proof. Choose a cover of S consisting of open sets $\{U_i\}$ for which any two points in U_i can be connected by a shortest geodesic.

Choose "enough" points in S so that they can all be connected by these geodesic segments.

Connect them by the geodesics. $\square$

We mention now

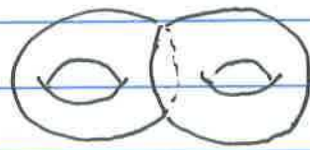
Cor 2.5.5 Let $(S, \{\Delta_i\})$ be a triangulated Riemann surface with $i = f$. Let e be the number of edges and v the number of vertices. Then

$$\chi(S) = f - e + v$$

is the Euler characteristic of S.

Rmk. Take this as a definition. This χ will appear later in a different context, but is the same Euler characteristic.

RE Compute the Euler characteristics of S^2, T^2, and the two-holed torus:

.

You should get

$$\chi(S^2) = 2$$
$$\chi(T^2) = 0$$
$$\chi(T^2 \# T^2) = -2$$

Verify it!

Final Remark (for today):

The Euler characteristic is a topological invariant; It does not depend on a choice of triangulation or metric on S.