Today we introduce the idea of geodesics, then use them to define the geodesic ~~ex~~ponential map.

Consider a curve $\gamma: [0,1] \to S$, S a metric Riemann surface, and γ smooth. The length of γ is

$$\ell(\gamma) = \int_0^1 \lambda(\gamma(t)) \, |\dot{\gamma}(t)| \, dt \qquad (*)$$

We define the energy functional along γ by

$$E(\gamma) = \frac{1}{2}\int_0^1 \lambda^2(\gamma(t)) \, \dot{\gamma}(t) \, \dot{\bar{\gamma}}(t) \, dt \qquad (**)$$

Thus $\quad \frac{1}{2}\ell(\gamma)^2 \leq E(\gamma)$

with equality precisely if $\lambda(\gamma(t))|\dot{\gamma}(t)| \equiv$ constant. In this case, we say that γ is parametrized by (or proportional to) arc length.

Thus, the minima of $\ell(\gamma)$ that are parametrized by arc length are precisely the minima of E.

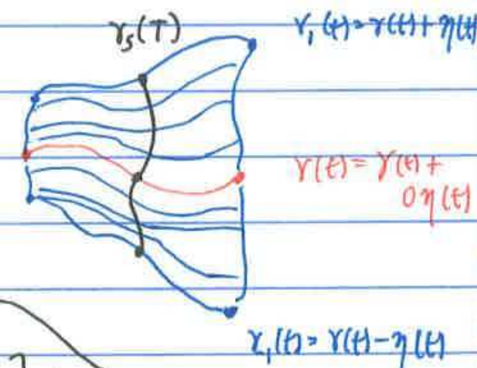
Thus E "selects a distinguished parametrization" for minimizers.

We want to characterize minimizers of E.

In local coordinates, we ~~put~~ let

$$\gamma(t) + s\eta(t) \qquad (***)$$

be a smooth variation of γ, $-s_0 \le s \le s_0$, $s_0 > 0$.



Fix $T \in (0,1)$.
Then
$\gamma_s(T) = \gamma(T) + s\eta(T)$
$: (-s_0, s_0) \to S$.

If γ minimizes E, we must have

$$0 = \frac{d}{ds}\left(E(\gamma + s\eta)\right)\Big|_{s=0} =$$

$$= \frac{1}{2}\int_0^1 \left[\lambda^2(\gamma)\left(\dot\gamma\dot{\bar\eta} + \dot{\bar\gamma}\dot\eta\right) + 2\lambda\left(\lambda_\gamma \eta + \lambda_{\bar\gamma}\bar\eta\right)\dot\gamma\dot{\bar\gamma}\right] dt$$

w/ $\lambda_\gamma = \frac{\partial\lambda}{\partial\gamma}$, etc.

$$= \mathrm{Re}\int_0^1 \left[\lambda^2(\gamma)\dot\gamma\dot{\bar\eta} + 2\lambda\lambda_\gamma\dot\gamma\dot{\bar\gamma}\bar\eta\right] dt$$

If the variation fixes the endpoints of γ, i.e., $\eta(0) = \eta(1) = 0$, we may integrate by parts to obtain

$$0 = -\mathrm{Re}\int_0^1 \left[\lambda^2(\gamma)\ddot\gamma + 2\lambda\lambda_\gamma\dot\gamma^2\right]\bar\eta\, dt$$

If this holds for all such variations η, we must have

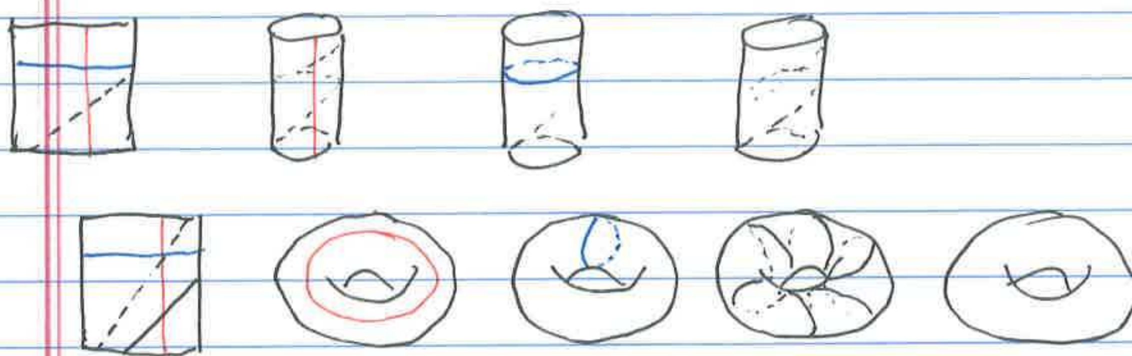
$$\ddot\gamma(t) + \frac{2\lambda_\gamma(\gamma(t))}{\lambda(\gamma(t))}\dot\gamma^2(t) = 0 \qquad (4*)$$

Def'n 2.3.9 A curve γ satisfying (4*) is called a geodesic of (S, g_λ).

Remark In the language of differential geometry, a geodesic is a curve satisfying $\nabla_{\dot\gamma}\dot\gamma = 0$, where $\dot\gamma$ is the velocity lift and ∇ is the Levi-Civita connection of the metric g_λ. Thus geodesics are curves w/ zero acceleration (in S).

Examples. 1.) straight lines are geodesics in $\mathbb{C}=\mathbb{R}^2$.

2.) By our discussion on Thursday, we can visualize the geodesics on the cylinder and torus.



(Lemma 2.3.7) 3.) Geodesics on $\mathbb{H}$ are circles (and great circles) orthogonal to the real axis of $\mathbb{C}$.

Pf. For the hyperbolic metric, (4*) becomes $\ddot{z}(t) + \frac{2}{z-\bar{z}}\dot{z}^2(t) = 0$. (RE)

for a curve $z(t)$ in $\mathbb{H}$. Writing $z(t) = x(t) + i\,y(t)$, we obtain

$$\ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0, \quad \ddot{y} + \frac{\dot{x}^2 - \dot{y}^2}{y} = 0 \qquad (RE)$$

If $\dot{x} = 0$, then x is constant and we obtain straight lines (great circles) intersecting the real line orthogonally.

If $\dot{x} \neq 0$, the first eqn becomes (yields)

$$\frac{\dot{x}}{y^2} = c_0,$$ so that $\dot{x} = c_0 y^2$ $\quad c_0 \neq 0$. (verify this!?)

Should this be $\ddot{x}$?

Jost says this is 0

Since all geodesics are parametrized by arclength, we have

$$\frac{1}{y^2}\left(\dot{x}^2+\dot{y}^2\right) = c_1^2 ,$$

we obtain
$$\left(\frac{\dot{y}}{\dot{x}}\right)^2 = \frac{c_1^2}{c_0^2\, y^2} - 1$$

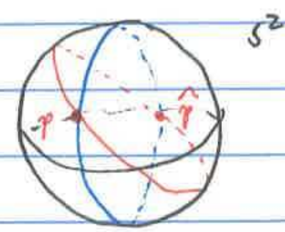
This equation is satisfied by circles

$$(x-x_0)^2 + y^2 = \left(\frac{c_1}{c_0}\right)^2 . \qquad \square$$

RE work out the details.

4.) The geodesics of the hyperbolic disk D are subarcs of circles and straight lines intersecting the boundary orthogonally.

Example. Any two geodesics on S^2 intersect each other at exactly 2 points that are diametrically opposite one another.



Projective 2-space, $P\mathbb{R}^2$ is defined by identifying these antipodal pts.
In this space, any two geodesics intersect in a single point. This is called (an) elliptic space.

Remark. Given a line (geodesic) l and a point $p_0 \notin l$, in elliptic geometry one cannot draw a parallel geodesic to l through p_0, since every geodesic through p_0 intersects l.
On the other hand, in hyperbolic geometry there exist infinitely many geodesics through p_0 that do not intersect l.

All other axioms of Euclidean geometry (save the parallel postulate) are valid in both elliptic and hyperbolic geometries.
Thus the parallel postulate is independent of the remaining axioms of (Euclidean) geometry.

This was discovered by Gauß, Bolyai, and Lobačevsky independently.

Now let (S, g_λ) be a compact conformal Riemann surface w/

$$g_\lambda = \lambda^2(z)\, dz\, d\bar{z}$$

(in local coordinates).
Recall (4*) the geodesic equation

$$\ddot{\gamma}(t) + \frac{2\lambda_\gamma(\gamma(t))}{\lambda(\gamma(t))}\dot{\gamma}^2(t) = 0 .$$

Splitting $\gamma(t)$ into its real and imaginary parts, (4*) constitutes a system of two ODE. We obtain (by Picard-Lindelöf)

Lemma 2.3.A.1. Let (S, g_λ) be a compact conformal Riemann surface w/ local coord chart $\varphi: U \to V \subset \mathbb{C}$, and $g_\lambda = \lambda^2(z)\, dz\, d\bar{z}$. Let $p \in V$, $v \in \mathbb{C} = T_pS$.
There exists $\varepsilon > 0$ and a unique geodesic $\gamma: [0, \varepsilon] \to S$ w/

$$\gamma(0) = p \quad \text{and} \quad \dot{\gamma}(0) = v$$

w/ γ smooth in p and v. $\square$

We denote this unique geodesic by $\gamma_{p,v}$.

If $\gamma(t)$ solves (4*), then so does $\gamma(\mu t)$ for $\mu \in \mathbb{R}$. Thus
$$\gamma_{p,v}(t) = \gamma_{p,\mu v}(t/\mu) \quad \text{for } \mu > 0,\ t \in [0, \varepsilon].$$

Since $\gamma_{p,v}$ is smooth and $\{v \in \mathbb{C} \mid \|v\|_p^2 := \lambda^2(p) v \bar{v} = 1\}$ is cpt in S, there exists $\varepsilon_0 > 0$ s.t. for any $v \in \mathbb{C}$ w/ $\|v\|_p = 1$, $\gamma_{p,v}$ is defined on the interval $[0, \varepsilon_0]$. Now it follows that for any $w \in \mathbb{C}$ w/ $\|w\|_p \leq \varepsilon_0$, $\gamma_{p,w}$ is defined on (at least) $[0,1]$.

Let $V_p := \{v \in \mathbb{C} = T_pS \mid \gamma_{p,v} \text{ is defined on } [0,1]\}$. Thus V_p contains the ball
$$\{w \in \mathbb{C} = T_pS \mid \|w\|_p \leq \varepsilon_0\}.$$

Def'n. The exponential map at $p \in S$ is

$$\exp_p : V \to S : v \mapsto \gamma_{p,v}(1)$$

(identifying points in $\varphi(U) = V$ w/ the corresponding pts in S.)

Lemma 2.3.A.2 $\exp_p$ maps a neighborhood of $0 \in T_pS$ $(= V_p)$ diffeomorphically onto some neighborhood of $p \in S$.

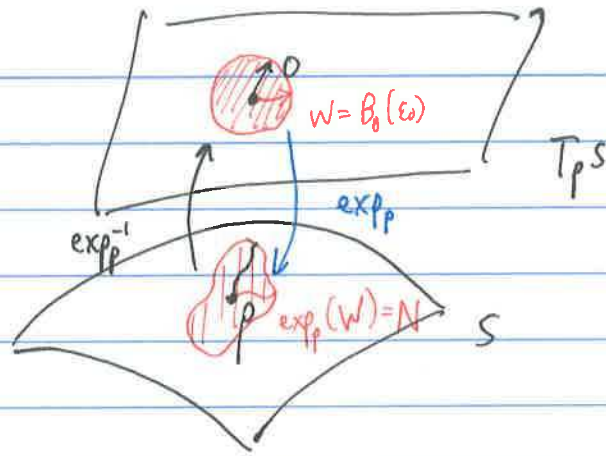
Proof. The derivative of $\exp_p$ at $0 \in T_pS$ applied to $v \in \mathbb{C}$ is
$$D(\exp_p(0))(v) = \frac{d}{dt} \gamma_{p,tv}(1)\Big|_{t=0}$$

$$= \frac{d}{dt} \gamma_{p,v}(t) \Big|_{t=0}$$
$$= \dot{\gamma}_{p,v}(0)$$
$$= v.$$

by the definition of $\gamma_{p,v}$. Thus $D(\exp_p(0)) = Id$. Apply the inverse function thm. □

However, the exponential map $\exp_p$ is in general not holomorphic. Thus, if we use $\exp_p^{-1}$ as a local chart, we are only guaranteed to preserve the smooth structure (not the conformal).

Thus, we need to investigate how our geometric objects transform under smooth coordinate transformations.
These will be REs. See pages 33–35 of JJ.



$\exp_p : W \to N$ is a diffeo. (not nec. holo.)

$(\exp_p|_N)^{-1} : N \to W$ may be regarded as a (smooth) local coordinate chart.
So-called exponential coordinates at $p \in S$.