Riemann Surfaces
Lecture 8

In groups, discuss the following examples.

Definition 2.3.8 The *hyperbolic* metric on H is the conformal metric on H given by $\lambda(z) = \frac{1}{y}$, where $z = x + iy$. The hyperbolic metric is thus given by

$$g_H = \frac{1}{y^2}\,dz\,d\bar{z}.$$

Example Find a potential function for g_H, compute the norm $\|\cdot\|_H$, and determine the curvature K_H. Show (or at least justify) that g_H is complete.

Lemma 2.3.6 The hyperbolic metric on D is given by $\lambda(z) = \frac{2}{1-|z|^2}$. Thus,

$$g_D = \frac{4}{(1-|z|^2)^2}\,dz\,d\bar{z}.$$

Proof The Möbius transformation

$$w(z) = \frac{z-i}{z+i}$$

is a conformal map from H to D sending the metric $\frac{1}{y^2}\,dz\,d\bar{z}$ to $\frac{4}{(1-|w|^2)^2}\,dw\,d\overline{w}$. Verify this claim.

Example The map $w(z) = e^{iz}$ is a local homeomorphism of H to the punctured disk $D \setminus \{0\}$, inducing the hyperbolic metric

$$g = \frac{1}{|w|^2\,(\log|w|^2)^2}\,dw\,d\overline{w}$$

on $D \setminus \{0\}$.

Finally, we consider the situation of HW 2.3,

Exercise 2.3.2 Consider the torus T^2 as constructed in chapter 1 with universal covering $\pi : \mathbb{C} \to T^2$ and group of covering transformations $H_\pi \cong \mathbb{Z}^2$. Let $g_\lambda = \lambda^2(z)dz\,d\bar{z}$ be a conformal metric on $\mathbb{C}$ such that H_π acts on $(\mathbb{C}, g_\lambda)$ by isometries. That is, each $\varphi \in H_\pi$ is an isometry of g_λ. Then g_λ induces a conformal Riemannian metric on T^2,

$$\overline{g}_\lambda([z]) = \lambda^2([z])\,dz\,d\bar{z}.$$

Moreover, every c.R. metric on T^2 lifts to a c.R. metric on $\mathbb{C}$ that is invariant under elements of H_π (verify this).

What then must be true about a c.R. metric on T^2, viewed as a c.R. metric on $\mathbb{C}$? To understand this, I recommend choose a "standardized" fundamental domain in $\mathbb{C}$ by setting $w_1 = 1$ and $w_2 = i$. Then view the the resulting parallelogram as $I^2 \subset \mathbb{R}^2$. This will also help you in computing

$$\int_{T^2} K_\lambda$$

which you should regard as

$$\int_{I^2} K_\lambda \, dA$$

in $\mathbb{R}^2$.