# Lecture 7

Begin w/ the cotangent bundle and 1-forms, and

$\boxed{\text{HW 2.1}}$ - Show that if $D_1, D_2$ are derivations, then so is their commutator $[D_1, D_2] = D_1 D_2 - D_2 D_1$. Thus $\mathcal{D}er_p(\mathcal{F}) = T_pS$ has a Lie algebra structure, as does $\mathfrak{X}(S) = \Gamma TS$.

**Defn.** A Riemannian inner product on a real vector space $V$ is a non-degenerate, symmetric, positive definite bilinear form on $V$.
That is, $\langle , \rangle : V \times V \to \mathbb{R}$ satisfies for all $x, y, z \in V$, $\alpha, \beta \in \mathbb{R}$,

i. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ (bi)-linearity

ii. $\langle x, y \rangle = \langle y, x \rangle$ symmetry

iii. $\langle x, y \rangle = 0$ for all $y \in V$ implies $x = \vec{0} \in V$ non-degeneracy

and iv. $\langle x, x \rangle \geq 0$ for all $x \in V$ and $\langle x, x \rangle = 0 \Leftrightarrow x = \vec{0}$ positive definite

A Riemannian inner product determines a norm $\|\cdot\|^2 : V \to \mathbb{R}$ on $V$, by $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$. (so $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$)

We can the use the (norm and) inner product to ~~do~~ define ~~lengths~~ distance and angles in $V$. Let $x, y \in V$. The distance between $x$ and $y$ is $d(x,y) = \|x - y\| \geq 0$. The angle between $x$ and $y$ is

$$\theta(x,y) = \theta := \cos^{-1}\left( \frac{\langle x, y \rangle}{\|x\| \, \|y\|} \right)$$

**Defn.** A Riemannian metric (tensor) on a manifold $S$ is an assignment of a Riemannian inner product $\langle , \rangle_p : T_pS \times T_pS \to \mathbb{R}$ in each tangent space that vary smoothly along $S$ (wrt $p$).

A Riemannian metric tensor may thus be considered as a section (sections are always smooth for us) of the bundle

$$\pi_{\otimes}^{*} : (TM \otimes TM)^{*} \longrightarrow M$$

that is symmetric and positive definite on each fiber.

We usually write $g$ to denote the metric (tensor) on the whole of $S$, but $g_p = \langle , \rangle_p$ in each fiber. (tangent space)

**Defn/Example** The dot product in $\mathbb{R}^n$ is a Riemannian inner product. If it is possible to define a metric $g$ on $S$ such that $g_p = \langle \rangle_p$ is the dot product for all $p$, then we say such a $g$ is an Euclidean metric on $S$.

NB! – Not every manifold admits an Euclidean metric.

Clearly, $\mathbb{R}^n$ and $\mathbb{C}^n$ admit an Euclidean metric for all $n$.

**Defn 2.3.1** A conformal Riemannian metric on a Riemann surface $S$ is given in local coordinates by

$$g_\lambda = \lambda^2(z)\, dz\, d\bar{z} \quad , \quad \lambda(z) > 0$$

w/ $\lambda \in \mathcal{F}(S) = C^\infty(S)$.

Recall $dz := dx + i\,dy$ and $d\bar{z} := dx - i\,dy$. We regard $dz$ and $d\bar{z}$ as "living" in the complexified cotangent bundle, $T^*_{\mathbb{C}}S = T^*S \otimes \mathbb{C}$ w/ fibers $T_p^*S \otimes \mathbb{C}$.

Now $dz\,d\bar{z} = (dx + i\,dy)(dx - i\,dy) = dx^2 + dy^2 = ds^2$, the arc length element of the Euclidean metric. Thus, a conformal Riemannian metric is a scaling in each tangent space of the Euclidean metric (dot product) on $T_pS$. In particular, the angles defined by $(g_\lambda)_p$ are exactly those defined by the dot product ($\boxed{\text{HW 2.2}}$). Hence the name "conformal" metric.

<u>Remark</u> Let $p \in S$ and consider two local coordinate charts $(U, z)$ and $(V, w)$ centered at $p$. Suppose $w \mapsto z(w)$ is the local coordinate transformation. Then the metric transforms to

$$g_\lambda = \lambda^2(z) \frac{\partial z}{\partial w} \frac{\partial \bar{z}}{\partial \bar{w}} \, dw\, d\bar{w}.$$

Let $z = x + iy$ and $w = u + iv$ s.t. $\frac{\partial}{\partial w} = \frac{\partial}{\partial u} - i\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial \bar{w}} = \frac{\partial}{\partial u} + i\frac{\partial}{\partial v}$

Let $|dz| = \sqrt{dx^2 + dy^2} = ds$.

The <u>length</u> of a rectifiable curve $\gamma : I \to S$ is

$$\ell(\gamma) := \int_\gamma \lambda(z)\, |dz|$$

The area of a measurable subset $B$ of $S$ is given by

$$\text{Area}(B) := \int_B \lambda^2(z) \, \frac{i}{2} \, dz \wedge d\bar{z}$$

Here $dz \wedge d\bar{z} = (dx + i\,dy) \wedge (dx - i\,dy) = dx \wedge dx - i\,dx \wedge dy + i\,dy \wedge dx + dy \wedge dy$

But $\wedge$ is alternating, so $dx \wedge dx = 0$ and $dy \wedge dx = -dx \wedge dy$.

thus $dz \wedge d\bar{z} = -2i\, dx \wedge dy$.

The distance between two points $z_1, z_2 \in S$ is then

$$d_\lambda(z_1, z_2) := \inf\{\ell_\lambda(\gamma) \mid \gamma : z_1 \mapsto z_2 \text{ is rectifiable}\}$$

**Def'n:** A metric is said to be complete iff every Cauchy sequence wrt $d_\lambda$ converges in $S$.

**Def'n 2.3.2** A potential for the metric $g_\lambda$ is a function $F : S \to \mathbb{R}$ such that

$$4 \frac{\partial^2}{\partial z \, \partial \bar{z}} F(z) = \lambda^2(z)$$

Recall that regarding the local coordinates in $\mathbb{R}^2$ ($\cong \mathbb{C}$), $\frac{\partial^2}{\partial z \, \partial \bar{z}}$ is the Laplacian.

**Lemma 2.3.1** Arc lengths, areas, and potentials do not depend on local coordinates. □

**Def'n 2.3.3** The Laplace-Beltrami operator wrt $g_\lambda$ is

$$\Delta_\lambda := \frac{4}{\lambda^2} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \overset{?}{=} \overset{4?}{\frac{1}{\lambda^2}} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) ?$$

I don't believe this. I will ~~think about it~~ and get back to you.

**Def'n 2.3.4** The curvature of $g_\lambda$ is $K = -\Delta \log \lambda$.

**Defn 2.3.5** A bijective map $h: S_1 \to S_2$ of Riemann surfaces w/ metrics $g_\lambda$ and $\bar{g}_\rho$ respectively is called an isometry iff it preserves angles and arc lengths.

**Better:** Let $TS_1$ and $TS_2$ be the tangent bundles of $S_1$ and $S_2$, and suppose $h: S_1 \to S_2$ is smooth. Then $h$ induces a map $h_*: TS_1 \to TS_2$ between tangent spaces

$$\begin{tikzcd} TS_1 \arrow[r, "h_*"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\circ"] & TS_2 \arrow[d, "\pi_2"] \\ S_1 \arrow[r, "h"'] & S_2 \end{tikzcd}$$

given in local $\mathbb{R}^2$ coordinates by the linearization $\frac{\partial h}{\partial(x,y)}$ (essentially the Jacobian at each point).

Then $h: S_1 \to S_2$ is an isometry if and only if

$$g_\lambda(x,y)_p = \bar{g}_\rho(h_* x, h_* y)_{h(p)}$$

for all $p \in S_1$ and all $x, y \in T_p S$.

**Lemma 2.3.2** $h = w(z)$ is an isometry iff it is conformal and

$$\rho^2(w(z)) \frac{\partial w}{\partial z} \frac{\partial \bar{w}}{\partial \bar{z}} = \lambda^2(z).$$

The potentials obey $F_1(z) = F_2(w(z))$. The Laplace-Beltrami operator and curvature are invariant under isometries as well.

The Euclidean metric is flat; that is, $K \equiv 0$.

<u>Lemma 2.3.3</u> Every compact Riemann surface $S$ admits a conformal Riemannian metric.

<u>Sketch Pf.</u> Choose $z \in S$ and a conformal chart $f_z : U_z \to \mathbb{C}$.
Choose a small open disk $D_z \subset f_z(U_z)$ and let
$\varphi_z = f_z|_{f_z^{-1}(D_z)} : U_z \cap f^{-1}(D_z) =: V_z \to \mathbb{C}$.

$S$ compact $\Rightarrow$ it can be covered by finitely many such nbds, $V_{z_i}, i = 1, \dots, m$.
For each $i$ choose $\eta_i : \mathbb{C} \to \mathbb{R}$ w/ $\eta_i > 0$ on $D_{z_i}$, $\eta_i = 0$ on $\mathbb{C} \setminus D_{z_i}$
w/ $\eta_i$ smooth.
($\eta_i$ is a bump function)

On $D_{z_i}$, define $g_i = \eta_i(w)\, dw\, d\bar{w}$. This induces local metrics on the $V_{z_i} = \varphi_{z_i}^{-1}(D_{z_i})$

Put $g_\lambda = \sum_{i=1}^{m} g_i$.

This $g_\lambda$ is smooth and positive on all of $S$, hence a conformal Riemannian metric. $\square$

<u>Example</u> Let $D := \{ z \in \mathbb{C} \mid |z| < 1 \}$ the open unit disk.
$H := \{ z \in \mathbb{C} \mid y = \operatorname{Im}(y) > 0 \}$ the (open) upper half plane.

For $z_0 \in D$, $\varphi_0(z) = \dfrac{z - z_0}{1 - \bar{z}_0 z}$ is a conformal self-map of $D$ sending $z_0$ to $0 \in D$.

Similarly, for $z_0 \in H$, $\psi_0(z) = \frac{z - z_0}{z - \bar{z}_0}$ is a conformal map of $H$ onto $D$ sending $z_0$ to $0 \in D$.

Thus $D$ and $H$ are conformally equivalent.

**Def'n 2.3.6** A Möbius transformation is a map $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ (or $S^2 \to S^2$) of the form

$$f : z \longmapsto \frac{az+b}{cz+d} \qquad a,b,c,d \in \mathbb{C}, \quad ad - bc \neq 0.$$

**Thm 2.3.2** Let $f : D \to D$ be holomorphic. Then for all $z_1, z_2 \in D$,

$$\left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)}\, f(z_2)} \right| \leq \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|, \qquad (*)$$

and for all $z \in D$, $$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}. \qquad (**)$$

Equality in either case implies $f$ is a Möbius transformation.

**Thm 2.3.3.** $$\left| \frac{f(z_1) - f(z_2)}{f(z_1) - \overline{f(z_2)}} \right| \leq \left| \frac{z_1 - z_2}{z_1 - \bar{z}_2} \right|, \quad z_1, z_2 \in H \qquad \text{and}$$

$$\frac{|f'(z)|}{\operatorname{Im}(f(z))} \leq \frac{1}{\operatorname{Im}(z)}, \qquad z \in H$$

under the same assumptions for a holomorphic self-map $f : H \to H$.

**Cor 2.3.1** Let $f : D \to D$ (or $f : H \to H$) be biholomorphic (conformal and bijective). Then $f$ is a Möbius transformation.