"Appendix" – Tangent and Cotangent Bundles – Differential forms

Let K be a commutative ring w/ identity and let A be a K-algebra. A derivation of A is a K-linear map $D: A \to A$ satisfying $D(ab) = D(a)b + aD(b)$ for all $a,b \in A$.

If D_1 and D_2 are two derivations, then so is their commutator $[D_1, D_2] = D_1D_2 - D_2D_1$. (RE)

Thus the space of all derivations of A, $Der_K(A)$, is a Lie algebra (also over K).

For any scalar $k \in K$, $D(k) = 0$. (RE)

Defn Let $\mathcal{F}(M) = C^\infty(M)$ denote the $\mathbb{R}$-algebra of smooth functions on M, and let $Der(\mathcal{F})$ be the space of derivations of $\mathcal{F}$.

Lie algebra!

A function $f: M \to \mathbb{R}$ is in $\mathcal{F}(M) =: \mathcal{F}$ if and only if $f \circ \varphi^{-1}: \mathbb{R}^n \to \mathbb{R}$ is infinitely differentiable for every chart φ.

Fix a point $p \in M$. WLOG let (x, U) be a chart @ p w/ local coordinates $(x) = (x^1, \ldots, x^n)$ s.t. $x(p) = \vec{0} \in \mathbb{R}^n$.

The partial derivatives $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ form a basis for $Der(\mathcal{F})$ at p.

We define $Der(\mathcal{F})_p = T_pM$ to be the tangent space at $p \in M$.

We now compare this def'n to the velocity vector def'n in our notes. Suppose $\gamma : (-a,a) \to M$ is a curve w/ $\gamma'(0)$ the tangent vector a $p = \gamma(0)$. Then the corresponding derivation is

$$D_\gamma(f) = (f \circ \gamma)'(0)$$

For each velocity $\gamma'(0)$ at p, $\gamma'(0) \mapsto D_\gamma$ is an isomorphism.

The tangent bundle of M is $TM = \bigsqcup_{p \in M} T_pM$.

The cotangent space to M at p is the dual space of T_pM, $T_p^*M = (T_pM)^*$. Recall that the dual space of a vector space is the space of all linear functionals on the vector space.

Let $dx^i \in T_p^*M$ be the linear functional such that $dx^i\left(\frac{\partial}{\partial x^i}\right) = 1$ and $dx^i\left(\frac{\partial}{\partial x^j}\right) = 0$ for $i \neq j$.

A section $\varphi : M \to T^*M$ w/ $\pi \circ \varphi = Id$, where $\pi : T^*M \to M$ is the bundle projection, is called a 1-form on M. A section $X : M \to TM$ w/ $\pi \circ X = Id$ is called a vector field on M.

Let $\mathfrak{X} = \mathfrak{X}(M)$ denote the space of all vector fields on M, and $\Omega^1(M)$ the space of 1-forms. Each $\varphi \in \Omega^1(M)$ is a map $\varphi : \mathfrak{X} \to \mathfrak{F}$.

Now we return to the notes to discuss metrics on M.