Riemann Surfaces
Lecture 6

We're finally ready to define a Riemann surface! First, a couple of preliminary definitions.

Definition holomorphic maps.

Let $f: \mathbb{C} \to \mathbb{C}$. f is complex differentiable at $z_0 \in \mathbb{C}$ iff the limit $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists. f is holomorphic iff it is complex differentiable at every z in its domain. (over)

Definition 2.1.1 surface.

A surface is a (connected) 2-dimensional manifold.

Definition 2.1.2 Riemann surface.

An atlas of charts $z_\alpha : U_\alpha \to \mathbb{C}$ on S is called conformal iff the transition functions $z_{\alpha\beta} = z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$ are holomorphic.

A conformal structure on S is a maximal conformal atlas.

A Riemann surface is a surface together w/ a conformal structure.

Remarks

Defn 2.1.3 A continuous map $h: S_1 \to S_2$ of Riemann surfaces is said to be holomorphic (or analytic) iff each local representative $z_\beta \circ h \circ z_\alpha^{-1}$ is holomorphic when it is defined.

A holomorphic map w/ non vanishing derivative $\frac{\partial h}{\partial z}$ is conformal.

We shall identify $U_\alpha \subset S$ w/ $z_\alpha(U_\alpha) \subset \mathbb{C}$, and $p \in S$ w/ $z(p) \in \mathbb{C}$, usually dropping the subscript.

Everything we study in this course will be local and invariant under change-of-charts. We should always prove this, but we'll frequently "believe" it, or leave it as an exercise.

holomorphic is sometimes identified w/ analytic. A function $f: \mathbb{C} \to \mathbb{C}$ is analytic iff it can be represented by a convergent power series,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for any $z_0 \in \mathrm{dom}(f) \subseteq \mathbb{C}$.

It is a fact that holomorphic is equivalent to analytic for complex functions.

In particular, every holomorphic function has infinitely many derivatives.

Let S be a Riemann surface.
Let $\mathcal{F}(S) = C^\infty(S)$ denote the algebra of smooth functions on S.
Define an equivalence relation on $\mathcal{F}$ by setting
f equivalent to g at $p \in S$ iff $\exists$ an open nbd $U \ni p$ s.t. $f|_U = g|_U$. The equivalence class $[f]_p$ of f at p is called the germ of f at p.

More remarks

Let S be a Riemann surface and $p \in S$. Let $I = (-a, a)$, $a > 0$, and consider the space of curves $\gamma : I \to S$ w/ $\gamma(0) = p$. Define an equivalence relation by $\gamma_1 \sim \gamma_2$ iff they have the same germ at p. Denote the set of all path germs by $\mathcal{P}_p$. Define an equiv. relation on $\mathcal{P}_p$ $[\gamma_1]_0 \sim [\gamma_2]_0$ iff

$$D(f \circ \gamma_1)(0) = D(f \circ \gamma_2)(0)$$

for all function germs $[f]_p$ at p. An equivalence class in $\mathcal{P}_p$ is a velocity vector at p and is denoted $[\gamma]_\sim$. <over>

Example 2.1.1 Trivial examples

$\mathbb{C}$ and open subsets of $\mathbb{C}$ are (non-compact) Riemann surfaces.
Any nonempty, open subset of a Riemann surface is again a Riemann surface.

Example 2.1.2 Riemann sphere

The sphere $S^2 \subset \mathbb{R}^3$ of Ch. 1 can be made into a Riemann surface by taking the same U_1 and U_2 but w/ charts

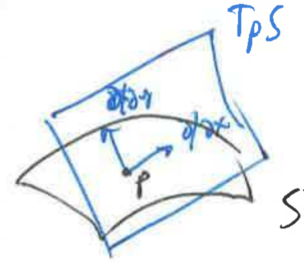
$$z_1 = \frac{x_1 + ix_2}{1 - x_3} \text{ on } U_1 \quad \text{and} \quad z_2 = \frac{x_1 - ix_2}{1 + x_3} \text{ on } U_2.$$

Then $z_2 = \frac{1}{z_1}$ on $U_1 \cap U_2$ so that the transition map is indeed holomorphic.

The Riemann sphere may also be realized as the extended complex plane $\mathbb{C} \cup \{\infty\}$ as follows.

Consider the map $z_1 = \frac{x_1 + ix_2}{1 - x_3}$ on all of S^2 (rather than just U_1).

Then, z_1 maps S^2 onto $\mathbb{C} \cup \{\infty\}$.



The (real) tangent space of S at p is $T_pS = \mathcal{P}_p/\sim$.

We have $T_pS \approx \mathbb{R}^2 \approx \mathbb{C}$ as a vector space.

The tangent bundle to (of) S is $TS = \coprod_{p \in S} T_pS$. It is a real manifold of dimension 4.

In a neighborhood of each $p \in S$, the tangent bundle has a product structure $TS|_U = U \times \mathbb{R}^2$ where $U \ni p$.

Let $\partial/\partial x$ and $\partial/\partial y$ be the basis for $\mathbb{R}^2$ corresponding to the coordinate directions.

A vector field on S is a map $S \to TS$ given locally by $X = f\,\partial/\partial x + h\,\partial/\partial y$, $f, h \in \mathcal{F}(S)$.

The space of all vector fields on S is $\mathfrak{X}(S)$.

Vector fields act like derivatives. For $g \in \mathcal{F}(S)$, $Xg = f\frac{\partial g}{\partial x} + h\frac{\partial g}{\partial y} \in \mathcal{F}(S)$.

Now, reincorporating the complex structure, put

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$$

Then $$\frac{\partial^2}{\partial z\,\partial \bar{z}} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta$$

* — See the "Appendix" on pages 26ff for more info and some REs.

Example 2.1.2, continued

$z_1(U_1) = \mathbb{C}$ and $z_1(U_2) = (\mathbb{C}\setminus\{0\}) \cup \{\infty\}$

Define $V_1 := \mathbb{C}$ and $V_2 := (\mathbb{C}\setminus\{0\}) \cup \{\infty\}$.

Then $\mathbb{C} \cup \{\infty\}$ is a Riemann surface w/ charts

$$\mathrm{Id} : V_1 \to \mathbb{C} \quad \text{and} \quad z \mapsto \frac{1}{z} \text{ on } V_2.$$

Now, a function h on a Riemann surface S is meromorphic iff every pt has a coordinate nbd s.t. either h or $\frac{1}{h}$ is holomorphic. This is equivalent to saying each pt has a nbd that maps holomorphically to either V_1 or V_2 in $\mathbb{C} \cup \{\infty\}$.

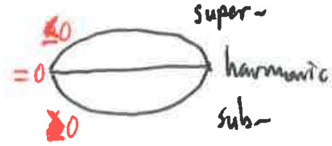
Example 2.1.3 $\mathbb{T}^2$

The torus, as defined in Ch.1 is a Riemann surface. The atlas is conformal. Indeed, the transition functions $\varphi_{\alpha\beta} : \Delta_\alpha \to \Delta_\beta$ are given by $\varphi_{\alpha\beta}(z) = z + m\omega_1 + n\omega_2$ for some $m, n \in \mathbb{Z}$. This is clearly holomorphic.

Definition (sub/super) harmonic functions

A function $f : S \to \mathbb{R}$ is (sub / super) harmonic iff in a local conformal coordinate,

$$\frac{\partial^2}{\partial z \partial \bar{z}} f \ \underset{\text{sub}}{(\geq} / \underset{\text{super}}{\leq)} = 0.$$



Lemma 2.2.1 *On a compact Riemann surface S, every subharmonic function (hence also every harmonic or holomorphic function) is constant.*

Sketch of proof

$f : S \to \mathbb{R}$ subharmonic.

S compact $\Rightarrow$ f attains its maximum at some point $p \in S$.

$z : U \to \mathbb{C}$ local chart w/ $p \in U$.

Then $f \circ z^{-1}$ is subharmonic on $z(U)$ and attains its maximum at an interior pt of $z(U)$. Therefore it is constant by the maximum principle. $\square$

→ Lemma 1.4.1 (over) first.

Lemma 2.2.2 *Let S be a simply connected surface, and $F : S \to \mathbb{C}$ a continuous function, nowhere vanishing on S. Then $\log(F)$ can be defined on S. That is, there exists a continuous function f on S with $e^f = F$.*

Sketch of proof

For every $p_0 \in S$, $F(p_0) \neq 0 \Rightarrow \exists$ open, connected $U \subseteq S$ s.t.
$\|F(p) - F(p_0)\| < \|F(p_0)\|$ for all $p \in U$.
Let $\{U_\alpha\}$ be the system (cover) of these nbds,
$(\log F)_\alpha$ a continuous branch of the logarithm of F in U_α,
and $F_\alpha = \{(\log F)_\alpha + 2n\pi i, n \in \mathbb{Z}\}$.
Then $\exists f : S \to \mathbb{C}$ s.t. $f|_{U_\alpha} = (\log F)_\alpha + n_\alpha 2\pi i$, $n_\alpha \in \mathbb{Z}$. (See Lemma 1.4.1 (over))
Then f is continuous and $e^f = F$.

Definition harmonic conjugates

Let $u : S \to \mathbb{R}$ be harmonic. A function $v : S \to \mathbb{R}$ is called a harmonic conjugate of u iff $(u + iv)$ is holomorphic.

Lemma 2.2.3 *Let S be a simply connected Riemann surface, and $u : S \to \mathbb{R}$ a harmonic function. Then there exists a harmonic conjugate to u on the whole of S.*

Sketch of proof

Choose U_α conformally equivalent to the disc D, and v_α a local harmonic conjugate of u in U_α. Let
$F_\alpha := \{v_\alpha + c \mid c \in \mathbb{R}\}$.
Then Lemma 1.4.1 $\Rightarrow \exists\, v : S \to \mathbb{R}$ s.t. for all α,

$$v|_{U_\alpha} = v_\alpha + c_\alpha, \quad c_\alpha \in \mathbb{R}.$$

Such a v is harmonic and conjugate to u. □

Lemma 1.4.1 –

Let M be a simply connected manifold and $\{U_\alpha\}$ an open cover of M. Assume all U_α are connected. Suppose on each U_α a family F_α of functions, w/ $F_\alpha \neq \emptyset$ for some α, satisfying

i) if $f_\alpha \in F_\alpha$, $f_\beta \in F_\beta$ and $V_{\alpha\beta}$ a component of $U_\alpha \cap U_\beta$, then ($f_\alpha \equiv f_\beta$ in a nbd of some $p \in V_{\alpha\beta}$. implies $f_\alpha \equiv f_\beta$ on $V_{\alpha\beta}$).

ii) $f_\alpha \in F_\alpha$ and $V_{\alpha\beta}$ a component of $U_\alpha \cap U_\beta$, $\Rightarrow$ there exists a function $f_\beta \in F_\beta$ w/ $f_\alpha \equiv f_\beta$ on $V_{\alpha\beta}$.

Then there exists a function f on M s.t. $f|U_\alpha \in F_\alpha$ for all α. Indeed, given $f_{\alpha_0} \in F_{\alpha_0}$, there exists a unique f w/ $f|U_{\alpha_0} = f_{\alpha_0}$.

We assume this w/out proof, but the proof is on pages 15–16 if you want to read it.