Riemann Surfaces
Lecture 5

We ended last week with the following.

Definition 1.3.4 Let $\pi_1 : \widetilde{M}_1 \to M$ and $\pi_2 : \widetilde{M}_2 \to M$ be coverings. $(\pi_2, \widetilde{M}_2)$ is said to *dominate* $(\pi_1, \widetilde{M}_1)$ if and only if there exists a covering $\pi_{21} : \widetilde{M}_2 \to \widetilde{M}_1$ with $\pi_2 = \pi_1 \circ \pi_{21}$. $(\pi_2, \widetilde{M}_2)$ is said to be *equivalent* to $(\pi_1, \widetilde{M}1)$ if and only if there exists a homeomorphism $\pi_{21} : \widetilde{M}_2 \to \widetilde{M}_1$ with $\pi_2 = \pi_1 \circ \pi_{21}$.

Let $\pi : \widetilde{M} \to M$ be a covering, $p \in M$, $q_1 \in \pi^{-1}(p)$, $\gamma : I \to M$ a loop with $\gamma(0) = \gamma(1) = p$, and $\widetilde{\gamma} : I \to \widetilde{M}$ the lift of γ with $\widetilde{\gamma}(0) = q_1$. By Corollary 1.3.1 (HW 1.2), if $\gamma \sim p$, then $\widetilde{\gamma}$ is closed and $\widetilde{\gamma} \sim q_1$.

Definition Let $G_\pi(q_1) = G(\pi, q_1) := \{[\gamma] \in \pi_1(M, p) \mid \widetilde{\gamma} : I \to M \text{ with } \widetilde{\gamma}(0) = \widetilde{\gamma}(1) = q_1\}$. That is, $G_\pi(q_1)$ is the set of all equivalence classes of loops in M based at p whose lifts beginning at $q_1 \in \pi^{-1}(p)$ are closed in $\widetilde{M}$.

Lemma 1.3.4 *$G_\pi(q_1)$ is a subgroup of $\pi_1(M, p)$.*

□

Suppose $q_2 \in \pi^{-1}(p)$, $q_2 \neq q_1$, and let $\widetilde{\gamma}$ be a path in $\widetilde{M}$ such that $\widetilde{\gamma}(0) = q_1)$ and $\widetilde{\gamma}(1) = q_2$. Then $\gamma := \pi(\widetilde{\gamma})$ is a loop at $p \in M$. If g is any loop at $p \in M$, then the lift of g starting at q_1 is closed precisely when the lift of $\kappa_\gamma(g) = \gamma g \gamma^{-1}$ is closed at q_2. Hence,

$$G_\pi(q_2) = [\gamma] \cdot G_\pi(q_1) \cdot [\gamma^{-1}].$$

Thus, $G_\pi(q_1)$ and $G_\pi(q_2)$ are conjugate subgroups of $\pi_1(M, p)$. Conversely, every conjugate subgroup of $G_\pi(q_1)$ can be obtained in this way. It is also easy to see that equivalence classes of coverings yield the same conjugacy classes of subgroups in $\pi_1(M, p)$.

This leads us to

Theorem 1.3.2 *$\pi_1(\widetilde{M})$ is isomorphic to G_π, and we obtain in this way a bijective correspondence between conjugacy classes of $\pi_1(M)$ and equivalence classes of coverings $\pi : \widetilde{M} \to M$.*

Sketch of the proof

Corollary 1.3.2 *If M is simply connected, then every covering $\pi : \widetilde{M} \to M$ is a homeomorphism.*

$\square$

Corollary 1.3.3 *If $G_\pi = \{1\}$ and $\pi : \widetilde{M} \to M$ is the corresponding covering, then $\pi_1(\widetilde{M}) = 1$, and a path $\widetilde{\gamma}$ in $\widetilde{M}$ is closed precisely when $\pi(\widetilde{\gamma})$ is closed and null-homotopic. Moreover, if $\pi_1(M) = \{1\}$, then $\widetilde{M} = M$.*

$\square$

Definition 1.3.5 The covering $\widetilde{M}$ of M with $\pi_1(\widetilde{M}) = \{1\}$ is called the *universal covering* of M.

Theorem 1.3.3 *Let $f : N \to M$ be a continuous map, and $\pi_M : \widetilde{M} \to M$, $\pi_N : \widetilde{N} \to N$ the universal coverings. Then there exists a lift $\widetilde{f} : \widetilde{N} \to \widetilde{M}$ of covering spaces.*

The diagram commutes.

Proof Apply Thereom 1.3.1 to $f \circ \pi_N$. □

Definition 1.3.6 Let $\pi : \widetilde{M} \to M$ be a local homeomorphism. Then a homeomorphism $\varphi : \widetilde{M} \to \widetilde{M}$ is called a *deck transformation* (or *covering transformation* in the book) iff $\pi \circ \varphi = \pi$.

The covering transformations of a covering form a group. (**RE**)

Lemma 1.3.5 *If $\varphi \neq Id$ is a covering transformation, then φ has no fixed point.*

Proof

It follows that if $\varphi_1, \varphi_2 \in H_\pi$ with $\varphi_1(q) = \varphi_2(q)$ for any single point $q \in \widetilde{M}$, then $\varphi_1 = \varphi_2$.

Definition 1.3.7 Let $G \subset H$ be groups. Then $N(G) := \{h \in H \mid h^{-1}Gh \in G\}$ is called the *normaliser* of G in H. If $N(G) = G$, then G is said to be a *normal subgroup* of H and we write $G \trianglelefteq H$.

Theorem 1.3.4 *For any covering $\pi : \widetilde{M} \to M$, the group H_π of covering transformations is isomorphic to $N(G_\pi)/G_\pi$. Thus, if $\pi : \widetilde{M} \to M$ is the universal covering of M, then*

$$H_\pi \cong \pi_1(M).$$

First, we note

Corollary 1.3.4 *Let G be a normal subgroup of $\pi_1(M, p)$ and $\pi : \widetilde{M} \to M$ the covering corresponding to G according to Theorem 1.3.2. Let $q_1 \in \pi^{-1}(p)$. Then, for every $q_2 \in \pi^{-1}(p)$, there exists precisely one covering transformation φ with $\varphi(q_1) = q_2$. This φ corresponds to $\pi(\widetilde{\gamma}) \in \pi(M, p)$, where $\widetilde{\gamma}$ is any path in $\widetilde{M}$ from q_1 to q_2.*

□

Now, we prove the theorem.

Proof of the theorem

Example 1.3.2 Covering spaces of the torus, $\mathbb{T}^2$.