Riemann Surfaces
Lecture 5

We ended last week with the following.

Definition 1.3.4 Let $\pi_1 : \widetilde{M}_1 \to M$ and $\pi_2 : \widetilde{M}_2 \to M$ be coverings. $(\pi_2, \widetilde{M}_2)$ is said to *dominate* $(\pi_1, \widetilde{M}_1)$ if and only if there exists a covering $\pi_{21} : \widetilde{M}_2 \to \widetilde{M}_1$ with $\pi_2 = \pi_1 \circ \pi_{21}$. $(\pi_2, \widetilde{M}_2)$ is said to be *equivalent* to $(\pi_1, \widetilde{M}1)$ if and only if there exists a homeomorphism $\pi_{21} : \widetilde{M}_2 \to \widetilde{M}_1$ with $\pi_2 = \pi_1 \circ \pi_{21}$.

Let $\pi : \widetilde{M} \to M$ be a covering, $p \in M$, $q_1 \in \pi^{-1}(p)$, $\gamma : I \to M$ a loop with $\gamma(0) = \gamma(1) = p$, and $\widetilde{\gamma} : I \to \widetilde{M}$ the lift of γ with $\widetilde{\gamma}(0) = q_1$. By Corollary 1.3.1 (HW 1.2), if $\gamma \sim p$, then $\widetilde{\gamma}$ is closed and $\widetilde{\gamma} \sim q_1$.

Definition Let $G_\pi(q_1) = G(\pi, q_1) := \{[\gamma] \in \pi_1(M, p) \mid \widetilde{\gamma} : I \to M \text{ with } \widetilde{\gamma}(0) = \widetilde{\gamma}(1) = q_1\}$. That is, $G_\pi(q_1)$ is the set of all equivalence classes of loops in M based at p whose lifts beginning at $q_1 \in \pi^{-1}(p)$ are closed in $\widetilde{M}$.

Lemma 1.3.4 *$G_\pi(q_1)$ is a subgroup of $\pi_1(M, p)$.*

□

Suppose $q_2 \in \pi^{-1}(p)$, $q_2 \neq q_1$, and let $\widetilde{\gamma}$ be a path in $\widetilde{M}$ such that $\widetilde{\gamma}(0) = q_1)$ and $\widetilde{\gamma}(1) = q_2$. Then $\gamma := \pi(\widetilde{\gamma})$ is a loop at $p \in M$. If g is any loop at $p \in M$, then the lift of g starting at q_1 is closed precisely when the lift of $\kappa_\gamma(g) = \gamma g \gamma^{-1}$ is closed at q_2. Hence,

$$G_\pi(q_2) = [\gamma] \cdot G_\pi(q_1) \cdot [\gamma^{-1}].$$

Thus, $G_\pi(q_1)$ and $G_\pi(q_2)$ are conjugate subgroups of $\pi_1(M, p)$. Conversely, every conjugate subgroup of $G_\pi(q_1)$ can be obtained in this way. It is also easy to see that equivalence classes of coverings yield the same conjugacy classes of subgroups in $\pi_1(M, p)$.

This leads us to

Theorem 1.3.2 *$\pi_1(\widetilde{M})$ is isomorphic to G_π, and we obtain in this way a bijective correspondence between conjugacy classes of $\pi_1(M)$ and equivalence classes of coverings $\pi : \widetilde{M} \to M$.*

Sketch of the proof

Let $\tilde{\gamma} \in \pi_1(\tilde{M}, q)$ and put $\gamma := \pi(\tilde{\gamma})$. $\gamma \in G_\pi$ since $\tilde{\gamma}$ is closed, and π maps homotopic curves to homotopic curves, so

$$\pi_*: \pi_1(\tilde{M}, q) \to G_\pi(q)$$

is an isomorphism. (Apply Lemma 1.2.3 and Cor 1.3.1).

Conversely, let G be a subgroup of $\pi_1(M,p)$, and let $\tilde{M}_G$ be the set of all equivalence classes $[\gamma]$ of paths in M w/ $\gamma(0)=p$, two paths being equivalent if $\gamma_1(1) = \gamma_2(1)$ and $[\gamma_1 \gamma_2^{-1}] \in G$.

Define a ~~projection~~ map by $\pi_G([\gamma]) = \gamma(1)$.

The rest of the proof defines a manifold structure on $\tilde{M}_G$ making $\pi_G : \tilde{M}_G \to M$ a covering.

You should read it! pp. 10-11.

Corollary 1.3.2 *If M is simply connected, then every covering $\pi : \widetilde{M} \to M$ is a homeomorphism.*

□

Corollary 1.3.3 *If $G_\pi = \{1\}$ and $\pi : \widetilde{M} \to M$ is the corresponding covering, then $\pi_1(\widetilde{M}) = 1$, and a path $\widetilde{\gamma}$ in $\widetilde{M}$ is closed precisely when $\pi(\gamma)$ is closed and null-homotopic. Moreover, if $\pi_1(M) = \{1\}$, then $\widetilde{M} = M$.*

□

Definition 1.3.5 The covering $\widetilde{M}$ of M with $\pi_1(\widetilde{M}) = \{1\}$ is called the *universal covering* of M.

Theorem 1.3.3 *Let $f : N \to M$ be a continuous map, and $\pi_M : \widetilde{M} \to M$, $\pi_N : \widetilde{N} \to N$ the universal coverings. Then there exists a lift $\widetilde{f} : \widetilde{N} \to \widetilde{M}$ of covering spaces.*

The diagram commutes.

$$\begin{tikzcd}
\widetilde{N} \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_N"'] & \widetilde{M} \arrow[d, "\pi_M"] \\
N \arrow[r, "f"'] & M
\end{tikzcd}$$

Proof Apply Thereom 1.3.1 to $f \circ \pi_N$. □

Definition 1.3.6 Let $\pi : \widetilde{M} \to M$ be a local homeomorphism. Then a homeomorphism $\varphi : \widetilde{M} \to \widetilde{M}$ is called a *deck transformation* (or *covering transformation* in the book) iff $\pi \circ \varphi = \pi$.

The covering transformations of a covering form a group. **(RE)**

Lemma 1.3.5 *If $\varphi \neq Id$ is a covering transformation, then φ has no fixed point.*

Proof
Let $\Sigma = \{q \in \widetilde{M} : \varphi(q) = q\}$. Let $q \in \Sigma$ and $\widetilde{U}$ a nbd of q s.t. $\pi : \widetilde{U} \to \pi(\widetilde{U}) = U$ is a homeo. Let $\widetilde{V} \subset \widetilde{U}$ s.t. $\varphi(\widetilde{V}) \subset \widetilde{U}$, w/ $q \in \widetilde{V}$. For $q' \in \widetilde{V}$, we have $\pi(\varphi(q')) = \pi(q') \in U$, hence $\varphi(q') = q'$ since both q' and $\varphi(q') \in \widetilde{U}$. Thus Σ is open. Since Σ is obviously closed, either $\Sigma = \emptyset$ or $\Sigma = \widetilde{M}$. In the latter case, $\varphi = Id$. □

It follows that if $\varphi_1, \varphi_2 \in H_\pi$ with $\varphi_1(q) = \varphi_2(q)$ for any single point $q \in \widetilde{M}$, then $\varphi_1 = \varphi_2$.

Definition 1.3.7 Let $G \subset H$ be groups. Then $N(G) := \{h \in H \mid h^{-1}Gh \in G\}$ is called the *normaliser* of G in H. If $N(G) = G$, then G is said to be a *normal subgroup* of H and we write $G \trianglelefteq H$.

Theorem 1.3.4 *For any covering $\pi : \widetilde{M} \to M$, the group H_π of covering transformations is isomorphic to $N(G_\pi)/G_\pi$. Thus, if $\pi : \widetilde{M} \to M$ is the universal covering of M, then*

$$H_\pi \cong \pi_1(M).$$

First, we note

Corollary 1.3.4 *Let G be a normal subgroup of $\pi_1(M,p)$ and $\pi : \widetilde{M} \to M$ the covering corresponding to G according to Theorem 1.3.2. Let $q_1 \in \pi^{-1}(p)$. Then, for every $q_2 \in \pi^{-1}(p)$, there exists precisely one covering transformation φ with $\varphi(q_1) = q_2$. This φ corresponds to $\pi(\widetilde{\gamma}) \in \pi(M,p)$, where $\widetilde{\gamma}$ is any path in $\widetilde{M}$ from q_1 to q_2.*

□

Now, we prove the theorem.

Proof of the theorem

Choose $p \in M$ and $q \in \pi^{-1}(p)$, and let $\gamma \in N(G_\pi(q))$. For any $y \in \widetilde{M}$, let $\tilde{\sigma} : [0,1] \to \widetilde{M}$ be a path joining q to y. Put $\sigma := \pi(\tilde{\sigma})$, and

$$\varphi_\gamma(q) = \widetilde{(\sigma \cdot \gamma)}(1).$$

If η is another path from q to y, then $\eta^{-1}\sigma \in G_\pi$, hence $\gamma^{-1}\eta^{-1}\sigma\gamma \in G_\pi$ since $\gamma \in N(G_\pi)$. Thus $\widetilde{(\eta \cdot \gamma)}(1) = \widetilde{(\sigma \cdot \gamma)}(1)$ and φ_γ does not depend on σ. We have,

$\pi(\varphi_\gamma(q)) = \pi(\widetilde{(\sigma \cdot \gamma)}(1)) = \pi(\tilde{\sigma}(1)) = \pi(q)$, so that φ_γ is a covering transformation. Moreover,

$$\varphi_{\gamma_2\gamma_1}(q) = \widetilde{(\gamma_2\gamma_1)}(1) = \varphi_{\gamma_2} \circ \varphi_{\gamma_1}(q),$$

hence $\varphi_{\gamma_2\gamma_1} = \varphi_{\gamma_2} \circ \varphi_{\gamma_1}$ by Lemma 1.3.5 and

$$\begin{aligned} \varphi_\gamma = Id &\Leftrightarrow \varphi_\gamma(q) = q \\ &\Leftrightarrow \tilde{\gamma}(1) = q \\ &\Leftrightarrow \gamma \in G_\pi. \end{aligned}$$

Thus, we have defined a morphism of $N(G_\pi)$ into H_π w/ kernel G_π.

Let $\varphi \in H_\pi$ and let $\tilde{\gamma}: [0,1] \to \tilde{M}$ be a path from q to $\varphi(q)$. Set $\gamma := \pi(\tilde{\gamma})$. Then $[\gamma] \in N(G_\pi)$ and $\varphi_\gamma(q) = \varphi(q)$. Hence $\varphi_\gamma = \varphi$ by Lemma 1.3.5. Thus "our" morphism is an epimorphism. $\square$

Example 1.3.2 Covering spaces of the torus, $\mathbb{T}^2$.

Work through this example together in class.

$\pi : \mathbb{C} \to \mathbb{T}^2$ is a covering and $\pi_1(\mathbb{C}) = \{1\}$, so $\mathbb{C}$ is the universal covering.

The covering transformations are

$$\varphi : z \longmapsto z + n w_1 + m w_2 \qquad n, m \in \mathbb{Z}.$$

Thus, $H_\pi \cong \mathbb{Z}^2$. By the Thm, we conclude, $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$.

$\mathbb{Z}^2$ abelian $\Rightarrow$ conj. subgroups are identical, therefore equivalence classes of coverings of $\mathbb{T}^2$ are in bijective correspondence w/ subgroups of $\mathbb{Z}^2$.

Consider the subgroup

$$G_{p,q} := \{ (pn, qm) \mid n, m \in \mathbb{Z} \} \qquad \text{for } p, q \in \mathbb{Z} \setminus \{0\}.$$

This group corresponds to the covering

$$\pi_{p,q} : T^2_{p,q} \to T^2$$

where $T^2_{p,q}$ is the torus generated by pw_1 and qw_2 as before.

By Thm 1.3.4, the group of covering transformations is

$\mathbb{Z}^2 / G_{p,q} \cong \mathbb{Z}_p \times \mathbb{Z}_q$. $(\alpha, \beta) \in \mathbb{Z}_p \times \mathbb{Z}_q$ acts on $T^2_{p,q}$ by

$$z \longmapsto z + \alpha w_1 + \beta w_2 .$$

The group $G_{1,0} = \{(n,0) \mid n \in \mathbb{Z}\}$ corresponds to the cylinder, and the group $G_{p,0} = \{(pn,0) \mid n \in \mathbb{Z}\}$, $p \neq 0$, corresponds to the cylinder C_p covering C.