We finish the lecture portion of the course w/ some applications of the Riemann-Roch Theorem.

First, a lemma that comes out of the proof of R-R.

<u>Lemma 5.4.4</u> For a canonical divisor K on a compact Riemann surface of genus p, we have

$$\deg K = \chi(S) = 2p-2.$$

<u>Sketch of pf.</u> Linearly equivalent divisors have the same degree since $\deg(g) = 0$ for g a meromorphic function.
All canonical divisors are linearly equivalent.

Let $V := \{$meromorphic 1-forms on S w/ no periods and residues, holomorphic on $S \setminus \bigcup\{z_\nu\}$ with poles of order ≤ 2 at the $z_\nu\}$.
This is a complex vector space.

Choose a nonconstant meromorphic function g on S with only simple poles. (One could integrate an $\eta \in V$ to achieve this.)
Put $K = (dg)$.

If $z \in S$ is not a pole of g, it is clear that

$$\operatorname{ord}_z(dg) = N_g(z)$$

where $N_g(z)$ is the order of ramification of g at z. If $g(z) = \infty$, then

$$\operatorname{ord}_z(dg) = \operatorname{ord}_z g - 1 = -2$$

since g has only simple poles. It follows that

$$\deg(dg) = \sum_{z \in S} N_g(z) - 2n$$

where n is the number of poles of g.

However, we may regard $g : S \to S^2$ as a holomorphic map of degree n and apply the Riemann-Hurwitz formula to get

$$\chi(S) = 2n - \sum_{z \in S} N_g(z). \qquad \square$$

We may also use R-R to prove the genus 0 case of the Uniformization Theorem:

Cor 5.4.1 Let S be a compact Riemann surface of genus 0. Then S is conformally equivalent to the unit sphere S^2.

Proof. We need to construct a holomorphic diffeomorphism $h : S \to S^2$. We shall use R-R to construct a meromorphic function g on S with a single simple pole and then interpret g as such a holomorphic diffeomorphism (onto S^2).

So, choose any $z_0 \in S$ and consider the divisor $D = z_0$.
Since $\text{genus}(S) = 0$, then $\deg K = -2$, so in particular $\deg(K-D) = -3 < 0$.
Thus $h^0(K-D) = 0$.
The Riemann-Roch Theorem thus yields $h^0(D) = 2$.

Therefore, we can find a nonconstant meromorphic function g with $D + (g) \geq 0$, i.e., with at most a simple pole at z_0.
As g is nonconstant, it must have a pole somewhere, and so it does at z_0.
Now, a meromorphic function $g: S \to \mathbb{C}$ may be regarded as a holomorphic map $g: S \to S^2$.
Since we have a simple pole at z_0, the mapping degree of h is 1.
By Riemann-Hurwitz, since the genuses of S and S^2 are both 0, h has no branch points, and being of degree 1, it must be a diffeomorphism. $\square$

We may also apply Riemann-Roch to compute the dimension on $Q(S)$ — the space of ~~qua~~ holomorphic quadratic differentials on $[S, f]$.

Cor 5.4.2 Let S be a compact Riemann surface of genus p, and let $Q(S)$ be the vector space of holomorphic quadratic differentials on S. Then

$$\dim_{\mathbb{C}} Q(S) = 0 \quad \text{if } p = 0,$$
$$\dim_{\mathbb{C}} Q(S) = 1 \quad \text{if } p = 1,$$
$$\dim_{\mathbb{C}} Q(S) = 3p - 3 \quad \text{if } p \geq 2.$$

Proof. Observe that $Q(S)$ can be identified w/ $\mathcal{L}(2K)$. Namely if $f dz$ is a 1-form with

$$(f dz) = K,$$

and if $g \in \mathcal{L}(2K)$, so that $(g) + 2K \geq 0$, then $g f^2 dz^2 \in Q(S)$. Conversely, if $\varphi dz^2 \in Q(S)$, then $g = \frac{\varphi}{f^2} \in \mathcal{L}(2K)$.

Now, $\deg(2K) = 4p-4$. Hence in $p=0$, we have

$$0 = h^0(2K) = \dim_{\mathbb{C}} \mathcal{L}(2K) = \dim_{\mathbb{C}} Q(S).$$

If $p=1$, then $\deg K = \deg 2K = 0$.

Also, since $p = \dim_{\mathbb{C}} H^0(S, \Omega^1) = 1$, there exists a holomorphic 1-form $f\,dz \not\equiv 0$ on S.

Since $\deg(f\,dz) = \deg K = 0$, $f\,dz$ cannot have any zeros. Hence $f^2 dz^2$ is nowhere zero on S. Hence, for any $\varphi \in Q(S)$, $\frac{\varphi}{f^2 dz^2}$ is a holomorphic function on S, hence a constant. It follows that $\dim_{\mathbb{C}} Q(S) = 1$.

Finally, let $p \geq 2$. Then $\deg(-K) = 2-2p < 0$, hence $h^0(-K) = 0$.

The Riemann-Roch Theorem now yields

$$h^0(2K) = 4p-4-p+1 = 3p-3.$$

We've already identified $\mathcal{L}(2K)$ w/ $Q(S)$, so we have the result. $\square$

<u>RE</u>. Define a holomorphic n-differential to be an object of the form

$$f(z)\,dz^n$$

with holomorphic f.

Determine the dimension of the space of holomorphic n-differentials on a compact Riemann surface of genus p.