<u>Def'ns</u> A meromorphic function $g \not\equiv 0$ is said to be a multiple of the divisor D iff

$$D + (g) \geq 0.$$

A divisor D is called a principal divisor iff it is the divisor of a meromorphic function $g \not\equiv 0$:

$$D = (g).$$

Two divisors D_1 and D_2 are linearly equivalent iff their difference is a principal divisor:

$$D_1 - D_2 = (g).$$

We define:

$\mathcal{L}(D) := \{ g : S \to \mathbb{C} \text{ meromorphic} \mid g \equiv 0 \text{ or } D + (g) \geq 0 \}$

this is a complex vector space

$h^0(D) := \dim_{\mathbb{C}} \mathcal{L}(D)$, and

$|D| := \{ D' \in \mathrm{Div}(S) \mid D' \geq 0,\ D' \sim D \}$

here $\sim$ denotes linear equivalence

<u>Rmks.</u> $g \not\equiv$ is holomorphic iff $(g) \geq 0$.

Any two canonical divisors are linearly equivalent: If $K = (\eta)$ and $K' = (\eta')$ are divisors of the meromorphic 1-forms η, η', then $\eta/\eta' =: g$ is a meromorphic function, and

$$K - K' = (g).$$

Finally, if S is compact, then

$$h^0(D) = \dim_{\mathbb{C}} |D| + 1$$

in the sense that $|D|$ can be identified with the projective space of the $h^0(D)$-dimensional vector space $\mathcal{L}(D)$. This is because if D' is linearly equivalent to D, then the meromorphic function g such that

$$D' = D + (g)$$

is unique up to a multiplicative constant ($\neq 0$), since the quotient of two such functions would be a nowhere vanishing holomorphic function on S, hence a (non-zero) constant.

Def'n. Let S be a compact Riemann surface, and $D \in \mathrm{Div}(S)$,

$$D = \sum s_\nu z_\nu .$$

The degree of D is defined as

$$\deg(D) := \sum s_\nu .$$

Clearly $\deg : \mathrm{Div}(S) \to \mathbb{Z}$ is a morphism of groups.

We have two Lemmas.

Lemma 5.4.1 For a meromorphic function $g \not\equiv 0$ on a compact Riemann surface S, $\deg(g) = 0$.

Lemma 5.4.2 Let S be compact. Then for $D \in \mathrm{Div}(S)$ with $\deg(D) < 0$, we have $h^0(D) = 0$.

And Now the central theorem on divisors

Thm 5.4.1 (Riemann-Roch) Let S be a compact Riemann surface of genus p, and D a divisor of S. Then

$$h^0(D) = \deg(D) - p + 1 + h^0(K - D).$$

Thus Riemann-Roch states that the number of linearly independent meromorphic functions g on S satisfying $(g) \geq -D$ equals [the degree of D − (genus of S) +1 + the number of linearly independent meromorphic 1-forms on S w/ $(\eta) \geq D$].

We should regard this theorem as an existence theorem for meromorphic functions w/ poles at most at those z_ν where $s_\nu > 0$. In particular, if $\deg D \geq p$, the theorem states that one can always find such a meromorphic function.

<u>Example</u>. [Genus 0 — The Riemann sphere.

Let P be any point in S and consider the divisor $n \cdot P$, $n \geq 0$. We consider the sequence $h^0(n \cdot P)$ of dimensions of $\mathcal{L}(n \cdot P)$ — functions that are holomorphic everywhere except at P, and have a pole of order at most n at P.
For $n=0$, $\mathcal{L}(0 \cdot P)$ is the space of entire functions on S. By Liouville's Thm, such functions must be constant. Thus $h^0(0 \cdot P) = 1$.

Let $P = \infty$ on the Riemann sphere. As we know, S^2 can be covered by two copies of $\mathbb{C}$ w/ transition function $z \mapsto 1/z$.
The one-form $\omega = dz$ on $\mathbb{C}$ extends to a meromorphic 1-form on the Riemann sphere; it has a double-pole at $P = \infty$ since

$$d\left(\frac{1}{z}\right) = -\frac{1}{z^2}\, dz.$$

Thus, its divisor is $K := \operatorname{div}(\omega) = (\omega) = -2P$.
letting $D_n = nP$, $n \geq 0$,

$$h^0(D_n) = \deg(D_n) - p + 1 + h^0(K - D_n) = n - 0 + 1 + h^0(-2P - nP)$$
$$= n+1$$

Now consider the torus.

As T^2 is flat, $T^2 = \mathbb{C}/M$, the one-form $\omega = dz$ is everywhere holomorphic on T^2. Thus $K = (\omega) = 0$.

R-R now gives,

$$h^0(n \cdot P) = \deg(n \cdot P) - p + 1 + h^0(K - n \cdot P)$$

$$= n - 1 + 1 + h^0(0 - n \cdot P)$$

$$= n + h^0(-n \cdot P)$$

when $n = 0$, $h^0(0 \cdot P) = 1$

otherwise, $h^0(n \cdot P) = n$.

<u>RE.</u> Compute the sequence for genus 2. Depending on the choice of base point, you'll get either

$1, 1, \boxed{1}, 2, 3, \ldots$

or $1, 1, \boxed{2}, 2, 3, \ldots$

The number in $\square$ is undetermined, but must be between 1 and 2, (inclusive) and an integer.

In fact, there are only 6 points (exactly 6) which have the second sequence.

Tomorrow - we end the class w/ some applications of Riemann-Roch.