Cor Let $\pi : \tilde{M} \to M$ be a covering, $\gamma : [0,1] \to M$ a loop w/ $\gamma(0) = \gamma(1) = p$, and $\tilde{\gamma} : [0,1] \to \tilde{M}$ a lift of γ. If γ is null-homotopic in M, then $\tilde{\gamma}$ is closed and null-homotopic in $\tilde{M}$.

Proof HW #3.

Defn. Let $\pi_1 : \tilde{M}_1 \to M$ and $\pi_2 : \tilde{M}_2 \to M$ be two coverings. $(\pi_2, \tilde{M}_2)$ is said to dominate $(\pi_1, \tilde{M}_1)$ if there exists a covering $\pi_{21} : \tilde{M}_2 \to \tilde{M}_1$ such that $\pi_2 = \pi_1 \circ \pi_{21}$.
Two coverings π_1 and π_2 are said to be equivalent if there exists a homeomorphism $\pi_{21} : \tilde{M}_2 \to \tilde{M}_1$ such that $\pi_2 = \pi_1 \circ \pi_{21}$.

Lemma $G_\pi := \{ [\gamma] \mid \tilde{\gamma} \text{ is closed} \}$ is a subgroup of $\pi_1(M, p)$.

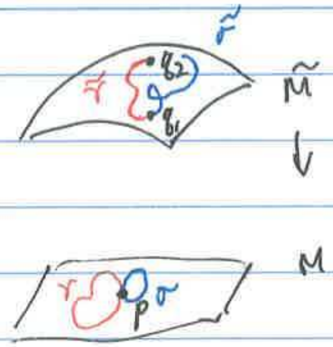
Remark. Clearly G_π depends on both the point $p \in M$ and a choice of $q \in \pi^{-1}(p) \subset \tilde{M}$. We write $G_\pi(q)$ to clarify, when necessary. There is no need to explicitly identify p as $p = \pi(q)$.

If $q_1, q_2 \in \pi^{-1}(p)$ and $\tilde{\gamma}$ is a path in $\tilde{M}$ from q_1 to q_2, then $\gamma := \pi(\tilde{\gamma})$ is a closed path in M w/ $\gamma(0) = \gamma(1) = p$.

If σ is a closed path at p, then the lift of σ starting at $q_1 \in \pi^{-1}(p)$ is closed precisely when the lift of $\sigma\gamma\sigma^{-1}$ starting at q_2 is closed. Hence,

$$G_\pi(q_2) = [\sigma] G_\pi(q_1) [\sigma^{-1}].$$

Thus $G_\pi(q_1)$ and $G_\pi(q_2)$ are conjugate subgroups of $\pi_1(M, p)$.



Conversely, every subgroup conjugate to $G_\pi(q_1)$ can be obtained in this way. Equivalent coverings yield the same conjugacy class of subgroups of $\pi_1(M,p)$.

Thm $\pi_1(\tilde{M})$ is isomorphic to G_π, and we obtain in this way a bijective correspondence between conjugacy classes of subgroups of $\pi_1(M)$ and equivalence classes of coverings $\pi: \tilde{M} \to M$.

* We'll start w/ this proof on Monday, leaving parts as exercises.

* We should finish w/ covering spaces on Monday or Tuesday, then we will define Riemann surfaces (finally!) and start talking about a little geometry. ☺