Let V be a vector space of dimension n. The <u>tensor algebra</u> of V is $T(V)$, the graded algebra of all k-fold tensor products of V, $k = 0, \dots,$ "∞".

$$T(V) = \bigoplus_{k=0}^{\infty} V^k$$

where $V^k = \underbrace{V \otimes \cdots \otimes V}_{k}$ and $V^0 = \mathbb{K}$, the underlying field.

Let I be the two-sided ideal consisting of elements $v \otimes v$. The <u>exterior algebra</u> of V is

$$\Lambda(V) = T(V)/I.$$

The exterior algebra inherits the grading of T w/

$\Lambda^0(V) = \mathbb{K}$

$\Lambda^1(V) = V$ and

$\Lambda^k(V) = 0$ for $k > n$.

In fact, $\dim(\Lambda^k(V)) = \binom{n}{k}$, so that $\Lambda^n(V) = \mathbb{K}$ as well.

We are interested in the case where M is a (real) smooth manifold of dimension n and V is a model cotangent space. We can create the external algebra bundle $\Lambda(T^*M) \to M$. (<u>RE</u> write out the details.)

<u>Defn</u>. A <u>differential k-form</u> on M is a smooth section of $\Lambda^k(T^*M) \to M$. We denote the space of all smooth k-forms on M by $\Omega^k(M) = \Omega^k$ if M is clear from context.

If $(x^1, \ldots, x^n)$ are local coordinates on M, then a differential form is an object of the form

$$\omega = \sum_{1 \le i_1 < \cdots < i_j \le n} \omega_{i_1 \cdots i_j} \, dx^{i_1} \wedge \cdots \wedge dx^{i_j}$$

j is called the degree of ω. Here $\wedge$ is the alternating product: $dx^i \wedge dx^k = \begin{cases} 0 & \text{if } i=k \\ -dx^k \wedge dx^i & \text{if } i \neq k \end{cases}$

The exterior derivative of ω is the form of degree $j+1$ defined by

$$d\omega := \sum_{i_0 = 1, \ldots, n} \; \sum_{1 \le i_1 < \cdots < i_j \le n} \frac{\partial}{\partial x^{i_0}} \omega_{i_1 \cdots i_j} \, dx^{i_0} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_j}$$

CBS — $d^2 = 0$.

A differential form ω is said to be closed if $d\omega = 0$, and exact if there exists a form α (of degree $j-1$) such that $\omega = d\alpha$.

Clearly, every exact form is closed.

Def'n. The j^{th} de Rham cohomology group of M is

$$H^j(M; \mathbb{R}) = \frac{\{\text{closed } j\text{-forms}\}}{\{\text{exact } j\text{-forms}\}}.$$

Here "j-form" means differential form of degree j.

We consider equivalence classes of degree j, where two closed forms ω_1 and ω_2 are equivalent iff there exists a $j-1$ form α such that $\omega_2 - \omega_1 = d\alpha$.
Thus $H^j(M;\mathbb{R})$ consists of equivalence classes closed j-forms that differ only by an exact j-form.
$H^j(M;\mathbb{R})$ is not only a group (under addition) but a real vector space.
d induces a morphism of groups $d: H^j(M;\mathbb{R}) \to H^{j+1}(M;\mathbb{R})$

We are interested in Riemann surfaces, so $n=2$. The only types of (real) differential forms are thus 0-forms (smooth functions), 1-forms (already studied), and 2-forms.
In local coordinates (x,y) a two form is represented by
$\Theta = \Theta(x,y)\, dx \wedge dy$ for some $\Theta \in \mathcal{F}(M) = \Omega(M)$.
As before, we put $dz = dx + i\,dy$ and $d\bar{z} = dx - i\,dy$ (locally).
Then $dz \wedge d\bar{z} = -2i\, dx \wedge dy$.

Now, suppose X is a surface, $[f] \in T(X)$ a complex structure on X, and $S = f(X)$. Let $\lambda^2(z)\, dz\, d\bar{z}$ be a conformal Riemannian metric.

The <u>fundamental two-form</u> or <u>Kähler form</u> of the metric is

$$\omega := \lambda^2(z)\, dx \wedge dy = \frac{i}{2} \lambda^2(z)\, dz \wedge d\bar{z}.$$

We define an operation $*$ on forms as follows.

For a complex-valued function $f: S \to \mathbb{C}$,

$$\star f := f\omega \qquad \text{or} \qquad \star f(z) = f(z)\lambda^2(z)\, dz \wedge d\bar{z}$$

For a one-form $\alpha = f\,dx + g\,dy$

$$\star\alpha := -g\,dx + f\,dy$$

or in complex notation, if $\alpha = u\,dz + v\,d\bar{z}$,

$$\star\alpha = -iu\,dz + iv\,d\bar{z}.$$

For a 2-form $\eta = h(z)\,dx \wedge dy$,

$$\star\eta := \frac{1}{\lambda^2(z)} h(z).$$

Notice $\star$ interchanges 0 and 2 forms, and maps 1-forms to 1-forms. Clearly $\star$'s action on 1-forms does not depend on the choice of metric. The others, however, do.

We can define a scalar product on the vector space of k-forms by

$$(\alpha_1, \alpha_2) := \int_S \alpha_1 \wedge \star\bar{\alpha}_2,$$

and thus obtain a Hilbert space

$$A_k^2 := \{k\text{-forms } \alpha \text{ w/ measurable coefficients and } (\alpha,\alpha) < \infty\}.$$

If α is a 1-form, for example, then in local coords

$$\alpha \wedge \star\alpha = i\,(u\bar{u} + v\bar{v})\, dz \wedge d\bar{z}$$
$$= 2\left(\|u\|^2 + \|v\|^2\right) dx \wedge dy$$

so that the product is indeed positive definite. Further

$$(\alpha_1, \alpha_2) = \bar{\lambda}\,(\alpha_2, \alpha_1)$$

and $$(\star\alpha_1, \star\alpha_2) = (\alpha_1, \alpha_2).$$

Thus, $\star$ is an isometry of A^2_k into A^2_{2-k}. It is also onto since $\star\star = (-1)^k$.

Further, if $\alpha_1 \in A^2_k$ and $\alpha_2 \in A^2_{k+1}$ are smooth and S is compact, then

$$(d\alpha_1, \alpha_2) = \int_S d\alpha_1 \wedge \star\bar{\alpha}_2$$

$$= (-1)^{k+1} \int_S \alpha_1 \wedge d(\star\bar{\alpha}_2) + \int_S d(\alpha_1 \wedge \star\bar{\alpha}_2)$$

$$= (-1)^{k+1} \int_S \alpha_1 \wedge d(\star\bar{\alpha}_2)$$

$$= -\int_S \alpha_1 \wedge \star(\star d \star \bar{\alpha}_2)$$

$$= -(\alpha_1, \star d \star \alpha_2)$$

We set $d^\star = -\star d \star$ and see that it is the (formal) adjoint of d wrt $(\cdot,\cdot)$.

<u>Note</u>: Even for 1-forms, $d^\star$ depends on the choice of metric.

Now let f be a smooth ~~complex-valued~~ function on S. We compute,

$$
\begin{aligned}
d^{*}d f &= d^{*}(\partial_z f\, dz + \partial_{\bar z} f\, d\bar z) \\
&= -{\star}d(-i\,\partial_z f\, dz + i\,\partial_{\bar z} f\, d\bar z) \\
&= -{\star}\left(2i \frac{\partial^2}{\partial z \partial \bar z} f \; dz \wedge d\bar z\right) \\
&= \frac{4}{\lambda^2} \frac{\partial^2}{\partial \bar z \partial z} f .
\end{aligned}
$$

$\star$

Thus $d^{\star}d$ is, up to a sign, the Laplace-Beltrami operator.

Defn's. A form α is said to be co-closed iff $d^{\star}\alpha = 0$, and co-exact there exists an η such that $\alpha = d^{\star}\eta$.

A 1-form α is said to be harmonic iff it is locally of the form $\alpha = df$ w/ f a harmonic function, and holomorphic iff it is locally $\alpha = dh$ w/ h holomorphic.

Lemma 5.2.1. $\alpha = u\,dz + v\,d\bar z$ is holomorphic iff $v \equiv 0$ and u hol. $\square$

Lemma 5.2.2. A 1-form η is harmonic iff $d\eta = 0 = d^{\star}\eta$. $\square$

Lemma 5.2.3 A 1-form η is harmonic iff it is of the form $\eta = \alpha_1 + \bar\alpha_2$, α_1, α_2 holomorphic.
A 1-form α is holomorphic iff it is of the form $\alpha = \eta + i{\star}\eta$, η harmonic.

We will need the more general notion of meromorphic 1-forms; i.e., one-forms that can be represented locally by

$$\eta(z) = f(z)\,dz$$

w/ f meromorphic.

We can consider the Laurent expansion about $z_0 = 0$ (in local coordinates)

$$f(z) = \sum_{n=n_0}^{\infty} a_n z^n .$$

The coefficient a_{-1} is called the residue of η at z_0:

$$\mathrm{Res}_{z_0} \eta := a_{-1} .$$

This number is independent of the choice of local chart, since

$$\mathrm{Res}_{z_0} \eta = \frac{1}{2\pi i} \int_\gamma \eta(z)$$

for any γ that is the boundary of a disk B containing z_0 in its interior, w/ η holomorphic on $\overline{B} \setminus \{z_0\}$.

Lemma 5.3.1 Let η be a meromorphic 1-form on a compact Riemann surface S, let $z_1, \dots, z_m$ be the singularities of η. Then

$$\sum_{j=1}^{m} \mathrm{Res}_{z_j} \eta = 0 .$$

Cor 5.3.1 A meromorphic function f on S has the same number (counting multiplicities) of zeroes and poles.

Classical terminology for Riemann surfaces: A differential form on S is of the

first kind iff η is holomorphic on S

second kind iff η is meromorphic, all of whose residues vanish

third kind iff η is meromorphic, all of whose poles are simple.

Onward toward Riemann-Roch.

Defn 5.4.1 Let S be a Riemann surface. A divisor on S is a locally finite formal linear combination
$$D = \sum s_\nu z_\nu \qquad (*)$$
with $s_\nu \in \mathbb{Z}$ and $z_\nu \in S$.

If S is compact, then D is finite.

The set of divisors on S forms an abelian group, denoted $\mathrm{Div}(S)$.

A divisor D is said to be effective iff $s_\nu \geq 0$ for all ν.

We write $D \geq D'$ if $D - D'$ is effective. Thus $D \geq 0$ means D is effective.

If $g \not\equiv 0$ is a meromorphic function on S, and $z \in S$, then $\mathrm{ord}_z g = k > 0$ if g has a zero of order k at z, and $\mathrm{ord}_z g = -k$ if g has a pole of order k at z. Otherwise $\mathrm{ord}_z g = 0$.

We define the divisor of the meromorphic function $g\ (\not\equiv 0)$ by
$$(g) = \sum (\mathrm{ord}_{z_\nu} g) z_\nu$$

If $\eta \not\equiv 0$ is a meromorphic 1-form, we can write $\eta = f dz$ locally and define $\mathrm{ord}_z \eta = \mathrm{ord}_z f$. We thus have $(\eta) = \sum (\mathrm{ord}_{z_\nu} \eta) z_\nu$, the divisor of η.

Defn. A canonical divisor, always denoted by K, is the divisor (η) of a meromorphic 1-form $\eta \not\equiv 0$ on S.