Quadratic Differentials

Let X be a topological surface, $[f] \in T(X)$ (or $\mathcal{T}(X)$), and $S = f(X)$ the resulting Riemann surface.

Recall that the cotangent bundle to X, denoted T^*X, is the bundle whose fibers are made up of all linear functionals on the corresponding tangent space. For any $p \in X$, T_p^*X is a vector space.

Over a coordinate nbhd $(U, (x,y))$, $T^*X|_U = T^*U$ is spanned by the elements dx and dy. Sections of T^*U are called local 1-forms and may be denoted $\theta = a(p)\,dx + b(p)\,dy$. Global sections of T^*X are called 1-forms.

The Riemann surface S has, in a sense, two cotangent bundles. The holomorphic cotangent bundle is spanned locally by the form $dz = dx + i\,dy$. The anti-holomorphic cotangent bundle is spanned locally by $d\bar{z} = dx - i\,dy$.

We write $T^{(1,0)*}S$ and $T^{(0,1)*}S$ to denote the holomorphic and anti-holomorphic cotangent bundles.

Together $T^{(1,0)*}S \oplus T^{(0,1)*}S = T^*_{\mathbb{C}}X = T^*X \otimes_{\mathbb{R}} \mathbb{C}$. is the complexified cotangent bundle of X. The bases $dz, d\bar{z}$ depend on the local coordinates $z = x+iy$ which in turn depend on the choice of $[f]$.

Let V be a vector space (over $\mathbb{R}$ or $\mathbb{C}$), and consider the tensor $V \otimes V$. Let I be the ideal generated by $v \otimes w - w \otimes v$, $v, w \in V$. The symmetric product of V with itself (the symmetric square) of V is $V \otimes V / I$.

Let $E \to X$ be a vector bundle w/ fiber V. The symmetric square of E is obtained by taking the symmetric square of each fiber. We write $\mathcal{S}(E) \to X$ to denote this bundle.

<u>Def'n</u>. A quadratic differential on a Riemann surface S is a section (smooth! only) of the bundle $\mathcal{S}(T^{(1,0)*}S) \to S$.
If the section is holomorphic, the it is a holomorphic quadratic differential.

Locally, in a conformal chart (U, z), a quadratic differential is of the form $\rho(z)\, dz^2 = \rho(z)\, dz \otimes dz$, where ρ is a complex-valued function on U (usually holomorphic for our purposes).

Thus, the local quad. diff. is holomorphic iff ρ is, and a global quad. diff. is holomorphic iff each local rep. is.

If $w \in \Omega^{(1,0)}(S)$ is a 1-form, then $w \otimes w = w^2$ is a quadratic differential.

<u>Singular Euclidean Structure</u>

A holomorphic quadratic differential q on S determines a Riemannian metric $|q|$ on the set $S \setminus Z$, where $Z = \{p \in S \mid q(p) = 0\}$.

On a local chart (U, z), if $q = \rho(z)\, dz^2$, then the associated Riemannian metric is $|\rho(z)|\, (dx^2 + dy^2)$.

Since ρ is holomorphic, the curvature of this metric is 0. Thus q defines a Euclidean metric on the complement of the zeros of its local rep.s ρ.

Suppose (U, z) and (V, w) are overlapping conformal charts w/ $w \mapsto z(w)$ conformal. Then $\varphi(z)\,dz^2$ pulls back to $\varphi(z(w)) \left(\frac{\partial z}{\partial w}\right)^2 dw^2$.

Recalling that $S = f(X)$, $[f] \in \mathcal{T}(X)$, we denote the set of holomorphic quadratic differentials on S by $Q(f)$.

RE. $Q(f)$ is a vector space. (over $\mathbb{C}$, hence also $\mathbb{R}$)

To be proved later, but worth stating now:

Cor 5.4.2 [JJ]

$\dim_{\mathbb{C}} Q(S^2) = 0$
$\dim_{\mathbb{C}} Q(T^2) = 1$
$\dim_{\mathbb{C}} Q(R_p) = 3p - 3$ for $p \geq 2$, where R_p is a Riemann surf. of type $(p, 0)$.
In particular, R_p is compact.

In particular, this means that there are no holomorphic quadratic differentials on the Riemann sphere.
And there is essentially just one (up to linear combinations) on T^2.

These dimensions should look familiar. In fact, we have

Thm 4.2.2 [JJ] (Teichmüller's Theorem)
Let X be a closed topological surface of type $(p, 0)$. Then $\mathcal{T}(X) = \mathcal{T}_p$ is diffeomorphic to $Q(f)$, where $[f]$ is any point in $\mathcal{T}_p$.

Proof: James's good problem. $\square$

We conclude today's short lecture by introducing another metric on Teichmüller space, from the point of view of quadratic differentials.

Let X be a surface of type $(p,0)$ (for now, but this extends to (p,n)), and consider a marking $[g] \in \mathcal{T}(X)$ as a Riemannian metric on X. In particular, $g = \lambda^2(z)\, dz\, d\bar{z}$. in local coords.

The natural L^2-metric on $Q(g)$ is given by

$$(\varphi_1 dz^2, \varphi_2 dz^2) := \int_{S=(X,g)} \varphi_1(z)\overline{\varphi_2(z)} \frac{1}{\lambda^2(z)} \frac{i}{2}\, dz \wedge d\bar{z} \qquad (*)$$

for $\varphi_1 dz^2, \varphi_2 dz^2 \in Q(g)$.

<u>Def'n 4.2.1</u> $(*)$ defines a Hermitian inner product on $Q(g)$ called the <u>Weil–Petersson product</u>.

The Weil–Petersson product gives rise to the <u>Weil–Petersson metric</u>, a Hermitian metric on $Q(g)$ which, by Teichmüller's Thm, may be regarded as a (~~Hermitian~~ Riemannian) metric on $\mathcal{T}_p$. (or $\mathcal{T}_{p,n}$ in general).

<u>Some properties</u>: The Weil–Petersson metric is Kähler on $\mathcal{T}_{p,n}$. It has negative holomorphic sectional, scalar, and Ricci curvatures. (Ahlfors, 1961 both results) It is usually not complete.