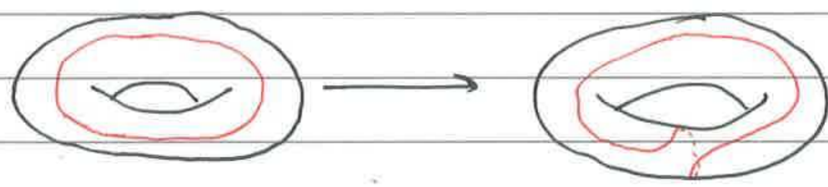
Wrapping up our study of Fricke spaces, we state a few more definitions and results.

Def'n. A Jordan curve on X is essential iff it is not null homotopic and it does not define an end of X.

Def'n. A homeomorphism of X onto itself is called a Dehn twist about an essential Jordan curve C on X iff it is homotopic to a homeomorphism that fixes all points in the complement of a neighborhood of C.

e.g.,



The homotopy classes of Dehn twists about C form an infinite cyclic group Δ_C.

Ex. Let m be an orientation preserving map of the closed annulus

$$A : \tfrac{1}{2} \le |z| \le 1$$

into X which takes the circle $|z|=1$ onto an essential curve C.

Let φ be the self-map of A given in polar coordinates by

$$r \mapsto r \qquad \theta \mapsto \theta + (4r-2)\varepsilon\pi \qquad (\tfrac{1}{2} \le r < 1,\ 0 \le \theta \le 2\pi)$$

where $\varepsilon = \pm 1$. Let M be a self-map of X which fixes every point of $m(A)$ and equals $m \circ \varphi \circ m^{-1}$ on $m(A)$.

Then M is a Dehn twist about C.

Such an M is called a primitive Dehn twist about C, a left (right) twist if $\varepsilon = -1$ $(+1)$.

The homotopy class of M generates Δ_C.

Lemma 8.1. Let $C_1, \ldots, C_k$ be disjoint essentially Jordan curves, none freely homotopic to another. Assume that a topological self-map w of X is homotopic to a product of Dehn twists about $C_1, \ldots, C_k$, but not null homotopic. Then the element w_* of $\mathcal{TM}(X)$ induced by w fixes no point of $\mathcal{T}(X)$.

Lemma 8.2. Two elements $[f_1], [f_2] \in \mathcal{T}(X)$ have the same image under the reduced Fenchel-Nielsen map Φ if and only if $[f_2] = w_*[f_1]$, where w is a product of Dehn twists about the partition curves $C_1, \ldots, C_d$ on X which define Φ.

The authors [BG] then construct a special fundamental polygon for $f(X) = \Gamma \backslash H$, and use it, and deformations of it, to show

Lemma 10.1 The reduced Fenchel-Nielsen Map is a local homeomorphism.

We already knew it was a continuous surjection, so now we know it is a covering map.

Read sections 9 and 10 for more details.

Section 11 is somewhat technical. It discusses the lengths of closed geodesics on X, and their behavior near funnels and ends.

We'll need

Def'n. A closed curve on a topological surface is called primitive iff it is not homotopic to another closed curve traversed more than once.

The main result is

Lemma 11.4. Let S be a Riemann surface of type (p,n) w/ $2p-2+n>0$. Let $L>0$, $\lambda_0>0$, and $0<\varepsilon<\frac{1}{2}$.
No primitive closed geodesic on S of length not exceeding L, and not an outer loop, can penetrate an ε-collar about an outer loop on S of length $\lambda\le\lambda_0$, or about a puncture.

This result is then interpreted in terms of the Fuchsian group for $S=\Gamma\backslash\mathbb{H}$.

It's worth noting that this follows in part from the known result (a Corollary in [BG]):

Cor. Let $L>0$ and S a Riemann surface of type (p,n), $2p-2+n>0$. Then there are only finitely many closed geodesics of length not exceeding L.

Section 12 puts a metric on Fricke space,

Theorem I. The Fricke space $\mathfrak{F}_{p,n}=\mathfrak{F}(X)$ has a metric $\delta=\delta_X$ which is consistent w/ the topology of $\mathfrak{F}(X)$ and which is invariant under the Fricke modular group $\mathfrak{FM}(X)$, as well as under any allowable map of $\mathfrak{F}(X)$. The δ-distance between two points $[f]$ and $[g]$ of $\mathfrak{F}(X)$ is

$$\delta([f],[g]) = \sup\left|\frac{1}{1+l_{[f]}(C)} - \frac{1}{1+l_{[g]}(C)}\right|$$

where C runs over all primitive closed curves on X.

RE. What properties does this metric have?

Section 13 studies the action of the Fricke modular group on Fricke space,

<u>Theorem II</u>. The Fricke modular group $\mathcal{FM}_{p,n} = \mathcal{FM}(X)$ acts properly discontinuously on the Fricke space $\mathcal{F}_{p,n} = \mathcal{F}(X)$.

"The result was stated by Fricke; his proof is hard to follow." [B,G]

The proof in [BG] is not hard to follow. They show that every isotropy subgroup of $\mathcal{FM}$ is finite, and every orbit of $\mathcal{FM}$ is discrete, They then apply a well known result of manifold theory,

<u>Lemma 13.3</u>. Let Γ be a group of isometries of a metric space M. If Γ has only finite isotropy groups and only discrete orbits, then Γ acts properly discontinuously on M.

Section 14 returns to the Fenchel-Nielsen coordinates. First,

<u>Proposition 14.1</u>. The reduced Fenchel-Nielsen map $\Phi: \mathcal{F}(X) \to (\mathbb{C}^*)^d \times (\mathbb{R}_+ \cup \{0\})^n$ is a universal covering.

~~The method of the proof is to show~~

<u>Proof</u>. Φ is surjective, continuous, and open (Lemmas 7.1 and 10.1). The subgroup $\mathcal{H}$ of $\mathcal{FM}(X)$ generated by Dehn twists about the partition curves acts freely (Lemma 8.1) and properly discontinuously (Th. II). Hence the natural map

$$\pi: \mathcal{F}(X) \to \mathcal{F}(X)/\mathcal{H}$$

is a Galois covering.

By Lemma 8.2, $\exists$ a bijection g such that $\Phi = g \circ \pi$. Since π is continuous and open, g is a homeomorphism, and Φ is a Galois covering w/ covering group $\mathcal{H}$.

Now $\pi_1(\Phi(^\circ\mathcal{I}(X))) = \pi_1((\mathbb{C}^*)^d \times (\mathbb{R}_+ \cup \{0\})^n) \cong \mathbb{Z}^d$ and so is the covering group $\mathcal{H}$ of Φ.
Here we can deduce from our study of Jost, ch1, that Φ is the universal covering, but [BG] continue,

Hence there is an exact sequence of covering groups

$$0 \to \Delta \to \pi_1 \to \mathcal{H} \to 0$$

or
$$0 \to \Delta \to \mathbb{Z}^d \to \mathbb{Z}^d \to 0$$

where Δ is (isomorphic to) the subgroup of π_1 defining Φ.
It follows that $\Delta = 0$ and Φ is universal. $\square$

Now, observe that the map
$$\exp: \mathbb{C}^d \times (\mathbb{R}_+ \cup \{0\})^n \to (\mathbb{C}^*)^d \times (\mathbb{R}_+ \cup \{0\})^n$$
defined by
$$(\xi_1, \dots, \xi_d, \rho_1, \dots, \rho_n) \mapsto (e^{\xi_1}, \dots, e^{\xi_d}, \rho_1, \dots, \rho_n)$$
is a universal covering.

Therefore, there exists a homeomorphism Ψ such that
$$\Phi = \exp \circ \Psi.$$
We say that Ψ is consistent (w/ the ordered partition of X used to define Φ) and we record

<u>Theorem III</u>. The Fricke space $\mathfrak{F}_{p,n}$ is homeomorphic to the product of $2d$ open and n half-open intervals. More precisely, an ordered maximal partition of the reference surface X induces a consistent homeomorphism Φ of $\mathfrak{F}_{p,n} = \mathfrak{F}(X)$ onto $\mathbb{C}^d \times (\mathbb{R}_+ \cup \{0\})^n$.

We call Φ the Fenchel-Nielsen map, and the numbers $\xi_1, \dots, \xi_d$, $\rho_1, \dots, \rho_n$ such that

$$(\xi_1, \dots, \xi_d, \rho_1, \dots, \rho_n) = \Phi^{-1}([f]), \quad [f] \in \mathfrak{F}(X),$$

the Fenchel-Nielsen coordinates of $[f]$.

And that does it for our study of Fricke and Teichmüller spaces.