Some definitions,

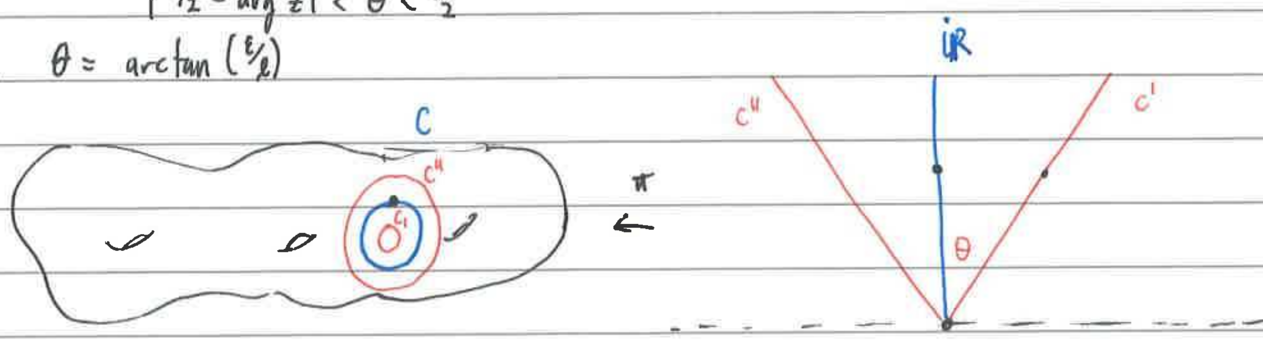
A (non-oriented) geodesic Jordan curve will be called a loop — an inner loop if it is not outer (i.e., if it does not define a hole).

Let C be any loop and $\varepsilon > 0$. An ε-collar about a loop C is a doubly connected subdomain of S, of area 2ε, bounded by two distinct Jordan curves C' and C'' that are equidistant from C. (C' and C'' must have the same constant distance from C.)

If C has length ℓ and is the image under π of the positive imaginary axis (we can always arrange for this via conjugation), then an ε-collar about C, if it exists, is the image under π of the sector

$$|\pi/2 - \arg z| < \theta < \frac{\pi}{2}$$

where $\theta = \arctan(\varepsilon/\ell)$



The length m of the Jordan curves C' and C'' (the images under π of the rays $\arg z = \frac{\pi}{2} \pm \theta$) is given by

$$m = \frac{\ell}{\cos\theta}$$

So that $m^2 = \ell^2 + \varepsilon^2$, and the distance δ from C' (or C'') to C is

$$\delta = \log\left(\frac{1+\sin\theta}{\cos\theta}\right)$$

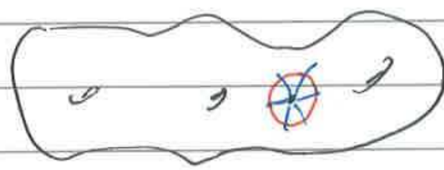
So that $\delta = \log\left(\sqrt{1+(\frac{\varepsilon}{\ell})^2} + \frac{\varepsilon}{\ell}\right)$.

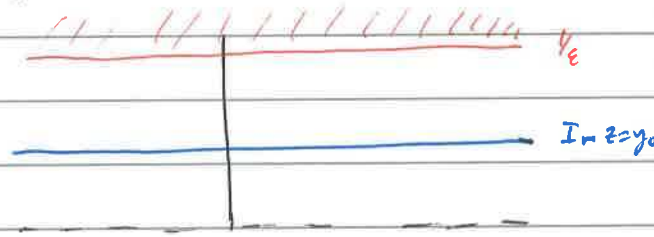
These formulas are valid independent of normalization; i.e., that $\pi^{-1}(C) = i\mathbb{R}$.

For future reference, for $\varepsilon_1 > \varepsilon_2 > 0$ this distance between the ε-collar boundaries is

$$\delta_{\varepsilon_1,\varepsilon_2} = \log\left(\frac{\sqrt{l^2+\varepsilon_1^2} + \varepsilon_1}{\sqrt{l^2+\varepsilon_2^2} + \varepsilon_2}\right).$$

An ε-collar about a puncture on S is a doubly connected domain in S of area ε bounded by the puncture and by a horocycle (i.e., a Jordan curve C orthogonal to the pencil of geodesics leading toward the puncture).





If the puncture is defined by the π-image of the line $\mathrm{Im}\, z = y_0 > 0$ and the corresponding maximal parabolic subgroup of G is generated by $z \mapsto z+1$, then the ε-collar, if it exists, is the image of the half plane $\mathrm{Im}\, z > \frac{1}{\varepsilon}$. We conclude that the horocycle bounding the ε-collar has length ε, and for $\varepsilon_1 > \varepsilon_2$ the distance between ε_1- and ε_2-collars about the same puncture is

$$\delta_{\varepsilon_1,\varepsilon_2} = \log\left(\frac{\varepsilon_1}{\varepsilon_2}\right).$$

This formulas remains valid for $l=0$ and $\delta = \infty$.

<u>Prop 4.1</u> (Collar Lemma) About every loop of length l and about every puncture (of length $l=0$, by defn) on a Riemann surface S of type (p,n) there is an ε-collar for every ε w/

$$0 < \varepsilon \leq \frac{l}{2}\operatorname{csch}\frac{l}{2}, \quad \text{regarding } 0 \cdot \operatorname{csch}(0) = 1.$$

Two such collars about distinct punctures or ~~holes~~ loops, or about a ~~hole~~ loop and a puncture, do not intersect.

This was proved for small l by Keen,[71] for all l by Halpern,[81] and the sharpness is due to Matelski (76).

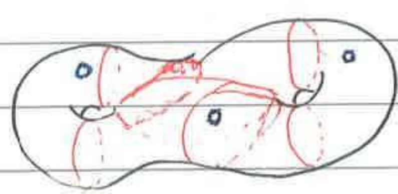
Now on to §7. <u>Reduced Fenchel-Nielsen Maps</u>

Let X be a topological space of type (p,n). A <u>maximal partition</u> of X (also called a pants decomposition) is a set of $d = 3p-3+n$ disjoint homotopically non-trivial Jordan curves (partition curves) on X, none of which is freely homotopic to another, and none of which defines an end on X.

A component of the complement of the partition is called a <u>region of the partition.</u> In our case there are $a = 2p-2+n$ regions, each a triply connected domain.

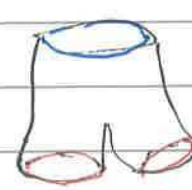
Each end of a region σ is a <u>bank</u> of a partition curve or of an end of X. If $(p,n) \neq (0,3)$, at least one bank of a region is the bank of a partition curve.

S of type $(2,3)$



"or something"

Each region looks like a pair of pants and has at least one "red" (partition) bank.



Next week – we'll state some results about triply connected domains, finish our study of Fricke space, and move on to quadratic differentials, divisors, and Riemann-Roch.