§3. We now construct a homeomorphism of $T_{p,n}$ into $\mathbb{R}^N$ for large enough N. Among other consequences, this implies that $T_{p,n}$ is Hausdorff.

Let $S = \Gamma \backslash H$ be a closed Riemann surface of genus $p \geq 2$.

Let $\gamma \in \Gamma$ and consider the (not-ordered) pair (γ, γ^{-1}).

Let ζ be any point in H and $\tilde{C}$ any curve in H joining ζ to $\gamma(\zeta)$. The canonical projection takes $\tilde{C}$ to a closed curve C on S.
Thus, we obtain our usual canonical bijection between conjugacy classes in Γ and free homotopy classes in $\pi_1(S)$.
We say γ and C correspond.

To every pair (γ, γ^{-1}), $\gamma \in \Gamma$, there corresponds a unique closed geodesic C on S, and vice versa. It is known that the absolute trace $|\tau|$ of γ and the length l of C are related by

$$\cosh \frac{l}{2} = \frac{|\tau|}{2}. \qquad \text{(RE)}$$

The construction of the desired mapping will depend on a choice of conformal structure on X — i.e., a pt in $T(X)$.

Once such a choice is made, choose a Fuchsian group G such that $X = G \backslash H$.

The choice of G can be made so that it contains elements g_1 and g_2 represented by matrices

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

$$B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{R}+), \quad \alpha+\beta = \gamma+\delta.$$

Let f be a homeomorphism of X onto a Riemann Surface. We have the commutative diagram

$$\begin{tikzcd} H \arrow[r, "\tilde{f}"] \arrow[d] & H \arrow[d] \\ X = G\backslash H \arrow[r, "f"'] & \tilde{f}G\tilde{f}^{-1}\backslash H = f(X) \end{tikzcd}$$

where the vertical arrows are holom. universal covering maps.

There is also an isomorphism $\quad g \in G \mapsto \chi_f(g) = \tilde{f} \circ g \circ \tilde{f}^{-1} \in \tilde{f} G \tilde{f}^{-1}$.

We may require then that $\chi_f(g_1)$ has fixed pts $\{0, \infty\}$, and $\chi_f(g_2)$ has a fixed pt at 1.

After a little more normalization, we obtain

Prop 3.1 Let $X = G\backslash H$, where G is a Fuchsian group containing two hyperbolic elements γ_1 and γ_2 w/ repelling fixed pts 0 and $\eta < 0$, and attracting fixed points ∞ and 1, resp. For every $[f]$ in $\mathcal{F}(X)$ there exists a canonical isomorphism

$$\chi_{[f]} : G \mapsto PSL(2, \mathbb{R})$$

such that

i) the repelling (attracting) fixed points of $\chi_{[f]}\gamma_1$ and $\chi_{[f]}\gamma_2$ are at 0 and $\eta' < 0$ (∞ and 1).

ii) $\chi_{[f]}G$ is a Fuchsian group and $\chi_{[f]}$ is induced by a lift to H of a homeomorphism of $X = G\backslash H$ onto $f(X) = \chi_{[f]}G\backslash H$, which belongs to $[f]$.

Prop 3.2 Under the same hypotheses, there is a finite set of elements $(\gamma_1, \dots, \gamma_N)$ in G such that the map

$$[f] \mapsto (|\operatorname{trace} \chi_{[f]}(\gamma_1)|, \dots, |\operatorname{trace} \chi_{[f]}(\gamma_N)|)$$

is a homeomorphism of $\mathcal{F}(X)$ into $\mathbb{R}^N$.

By the previous relation, $|\operatorname{trace} \chi_{[f]}\gamma_i| = 2\cosh \frac{1}{2} l_{[f]}(C_i)$.

The number $l_{[f]}(C_i)$ is a continuous function of $[f] \in \mathcal{F}(X)$ by defn of the topology.

The inverse map to the coordinate map is also continuous since

$$l_{[f]}(C) = 2 \operatorname{arc\,cosh} \tfrac{1}{2} |\operatorname{trace} x_{[f]}\, \gamma|$$

depends continuously on the Möbius transformations, $x_{[f]}(\gamma_i)$.

<u>Prop 3.3</u> There is a finite set of closed curves $C_1, \dots, C_N$ on X such that the map

$$[f] \longmapsto (l_{[f]}(C_1), \dots, l_{[f]}(C_N))$$

is a homeomorphism of $\mathcal{T}(X)$ into $(\mathbb{R}_+ \cup \{0\})^N$.

<u>Rmk</u>. This prop can be improved in two ways. The curves $C_1, \dots, C_N$ may be taken to be simple closed curves. Moreover, the element $[f] \in \mathcal{T}(X)$ is uniquely determined by the projective class of the vector

$$(l_{[f]}(C_1), \dots, l_{[f]}(C_N)).$$

§4. Punctures, Holes, Collars

Let $[f] \in T(X)$, and $f(X) = S = \Gamma \backslash H$.

A Jordan curve C on X defines an end B of X iff $X \backslash C$ has two components, precisely one of which is doubly connected.

If so, $f(C)$ defines an end $f(B)$ of $f(X) = S$.

If the Jordan curve is freely homotopic to C', then C and C' define the same end.

Intuitively, the ends of X represent the connected components of the "ideal boundary" of the space. Each end represents a topologically distinct way to move to infinity within the space.

Another defn.

Let $K_1 \subset K_2 \subset K_3 \subset \cdots$ be an ascending sequence of compact subsets of X whose interiors cover X.
Every sequence $U_1 \supset U_2 \supset U_3 \supset \cdots$, U_n a connected component of $X \backslash K_n$, is ~~an~~ an end of X.

As our space X was assumed to be of type (g, n), then X has n ends.

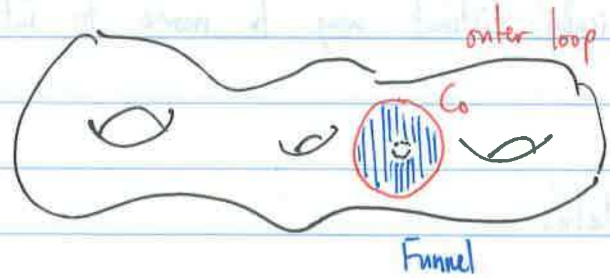
Now let C be as above. The <u>radius</u> of the end $f(B)$ is the number

$$r = \frac{1}{2\pi} l_{[f]}(C).$$

$f(B)$ is called a <u>hole</u> if $r > 0$ or a <u>puncture</u> if $r = 0$.

Every hole on $f(X)$ is defined by a unique geodesic Jordan curve C_0 of length $2\pi r$. We call C_0 the <u>outer loop</u> belonging to the end, and we call the doubly connected component of the complement $S \setminus C_0$ the <u>funnel</u> adjacent to C_0.

Ex.



$S_{3,1}$ w/ hole.

If F denotes the funnel, notice that $S_{3,1} \setminus F \cong S_{3,1}$ topologically (it's of type $(3,1)$). In general the complement of F has the same topological type (g, n) as S.

Two distinct funnels (or outer loops) are disjoint.

The complement in S of all funnels is called the Nielsen core of S and will be denoted by $\mathbb{K}(S)$.

The set $\mathbb{K}(S)$ is convex (wrt the Poincaré metric on S). It is compact iff S has no punctures.

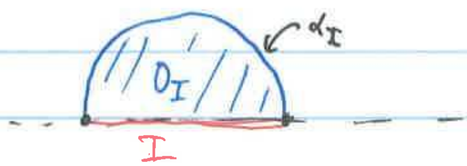
The limit set of G (the Fuchsian group) is the set Λ of accumulation points of orbits $\{g(z)\}$, z fixed, $g \in G$.

Λ either coincides w/ $\mathbb{R}_* = \mathbb{R} \cup \{\infty\}$ or is a closed, nowhere dense subset of $\mathbb{R}_*$.

The first case happens iff S has no holes.

In the second case, let I be a component of $\mathbb{R}_* \setminus \Lambda$. Thus I is an open interval and the stabilizer G_I of I in G is a cyclic group Γ_I generated by a hyperbolic $r \in G$ which fixes the endpoints of I.

Let O_I denote the domain in H bounded by I and the Poincaré line α_I in H w/ the same endpoints as I



The canonical projection $\pi: H \to G\backslash H$ sends α_I to the outer loop and O_I onto the funnel adjacent to this loop. All funnels are obtained in this way,

$$\pi^{-1}(\mathbb{K}(S)) = H \setminus \bigsqcup O_I$$

where I runs over all components of $\mathbb{R}_\infty \setminus \Lambda$.

A funnel F adjacent to an outer loop of length l, together w/ the restriction to F of the Poincaré metric on S, is uniquely determined by the number l.

Indeed, $C \cup F$ is isometric to the subdomain $0 < \arg z \leq \pi/2$ of H by the group generated by $z \mapsto e^l z$.

<u>CBS</u> — Every Riemann surface S of type (p,n), $2p-2+n > 0$, is the Nielsen core of a uniquely determined Riemann surface S'. S' has the same topological type as S and the same number of punctures.

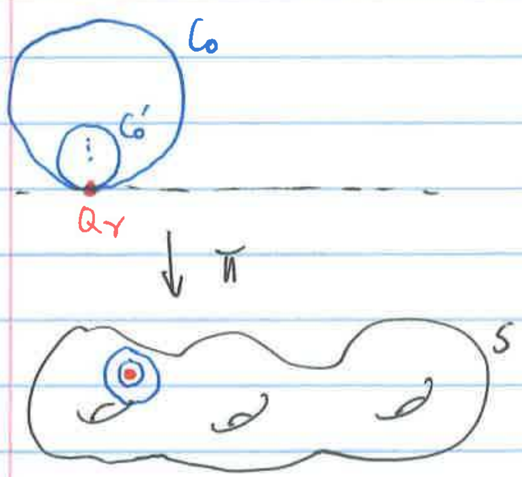
We call S' the <u>Nielsen extension</u> of S and denote it by $\mathbb{N}(S)$. If S has no holes, then $\mathbb{K}(S) = \mathbb{N}(S) = S$.

The punctures of S are in canonical bijection w/ conjugacy classes (in G) of maximal parabolic subgroups of G.

More precisely, let $\gamma \in G$, and Γ_γ be the subgroup generated by γ. Let Q_γ be the fixed point of γ.

Then the puncture on S corresponding to Γ_γ can be defined by $\pi(C_0)$ where C_0 is a suitably chosen horocycle at Q_γ. — i.e., a circle tangent to $\mathbb{R}$ at Q_γ and lying, except for the point Q_γ, entirely in H. If $Q_\gamma = \infty$, the horocycle is consider to be a horizontal line.

WLOG, let $\gamma : z \mapsto z+1$, so that every circle $\operatorname{Im} z = y_0 > 1$ is suitable.



$\pi(Q_\gamma) \notin S$

— it's the puncture