Fricke spaces

Our goal is to study Fricke spaces of surfaces that are homeomorphic to a closed surface of genus p with n distinct points removed. Such a surface is said to be of topological type (p,n). For this, we will follow Bers-Gardiner.

First, we study the case when R is a closed Riemann surface of genus g (with no points removed). For this, we continue to follow [IT, § 2.5].

Recall that Teichmüller space T_g, $g \geq 2$, consists of all marked Riemann surfaces $[R, \Sigma]$ where $\Sigma = \{[A_j],[B_j]\}_{j=1}^{g}$ is a canonical system of generators of $\pi_1(R,p) \cong \mathbb{Z}^{2g}$.

Recall further that $R = \Gamma \backslash H$ for some discrete subgroup Γ of $\mathrm{Aut}(H) \cong PSL(2,\mathbb{R})$. There is an isomorphism between Γ and $\pi_1(R,p)$. Denote by α_j and β_j the elements of Γ corresponding to $[A_j]$ and $[B_j]$, resp.

The subgroup Γ is called a Fuchsian model for R.

Fuchsian models have the ambiguity caused by inner automorphisms: for any $\delta \in \mathrm{Aut}(H)$, $\delta\Gamma\delta^{-1}$ is another Fuchsian model for the same R. Thus, we impose normalization conditions:

i) β_g has its repelling and attractive fixed points at 0 and ∞, respectively.
ii) α_g has its attractive fixed point at 1.

Remark. α_g and β_g must be hyperbolic, as we studied previously. α_g, β_g not commutative implies $Fix(\alpha_g) \cap Fix(\beta_g) = \emptyset$. The normalization conditions can be ensured by conjugation in $Aut(\mathbb{H})$.

Proposition 2.23
For a given marking Σ on a closed Riemann surface R of genus $g \geq 2$, a canonical system of generators $\{\alpha_j, \beta_j\}_{j=1}^{g}$ of a Fuchsian model Γ of R satisfying the normalization conditions wrt Σ is uniquely determined by the point $[R, \Sigma]$ in T_g.

We call this the normalised Fuchsian model of $[R, \Sigma]$.

Its generators $\{\alpha_j, \beta_j\}_{j=1}^{g}$ satisfy the sole relation

$$\prod_{j=1}^{g} [\alpha_j, \beta_j] = id,$$

$[\alpha_j, \beta_j] = \alpha_j \circ \beta_j \circ \alpha_j^{-1} \circ \beta_j^{-1}$.

Proof. Let (R,Σ) and (R',Σ') such that $[R,\Sigma]=[R',\Sigma'] \in T_g$.
Then $\exists$ $f: R \to R'$ biholomorphic s.t. $f_*(\Sigma)$ is equivalent to Σ'.
A lift $\tilde{f}$ of f to H is an element of $\mathrm{Aut}(H)$. Choose the lift s.t.

$$\alpha_j' = \tilde{f} \circ \alpha_j \circ \tilde{f}^{-1} \quad \text{and} \quad \beta_j' = \tilde{f} \circ \beta_j \circ \tilde{f}^{-1}$$

where $\{\alpha_j', \beta_j'\}$ are normalized generators of the Fuchsian model for R'.
By condition i), $\tilde{f}(z) = \lambda(z)$ for some $\lambda > 0$. By condition ii), α_g and α_g' have the same fixed point at 1. Hence $\lambda = 1$.
Thus $\tilde{f} = id$ and $\alpha_j' = \alpha_j$, $\beta_j' = \beta_j$. □

Lemma 2.24 Let $\{\alpha_j, \beta_j\}_{j=1}^{g}$ be the canonical system of generators of the normalized Fuchsian model Γ for a point $[R,\Sigma]$ in T_g. If an element $\gamma(z) = \frac{az+b}{cz+d}$ of $\{\alpha_j, \beta_j\}$ does not coincide w/ β_g, then $bc \neq 0$.

Proof. Suppose $b=c=0$. Then $\mathrm{Fix}(\gamma) = \mathrm{Fix}(\beta_g) = \{0, \infty\}$, and hence γ and β_g commute, a contradiction.
Now suppose $b=0$ and $c \neq 0$. Then $\mathrm{Fix}(\gamma) \cap \mathrm{Fix}(\beta_g) = \{0\} \neq \emptyset$ implies that Γ is not Fuchsian, a contradiction.
Similarly for $b \neq 0$, $c = 0$. □

By the Lemma, the canonical system $\{\alpha_j, \beta_j\}$ of generators of the normalized Fuchsian model Γ for $[R, \Sigma] \in T_g$ is written uniquely in the form

$$\alpha_j = \frac{a_j z + b_j}{c_j z + d_j} \qquad a_j, b_j, c_j, d_j \in \mathbb{R}, \quad c_j > 0, \quad a_j d_j - b_j c_j = 1$$

and

$$\beta_j = \frac{a_j' z + b_j'}{c_j' z + d_j'} \qquad a_j', b_j', c_j', d_j' \in \mathbb{R}, \quad c_j' > 0, \quad a_j' d_j' - b_j' c_j' = 1$$

for each $j = 1, 2, \ldots, g-1$.

Equivalently, $\alpha_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in PSL(2, \mathbb{R})$ w/ $c_j > 0$, and similarly for β_j, $j = 1, \ldots, g-1$.

Def'n. The Fricke coordinates $\mathcal{F}_g : T_g \to \mathbb{R}^{6g-6}$ are given by

$$\mathcal{F}_g([R, \Sigma]) = (a_1, c_1, d_1, a_1', c_1', d_1', \ldots, a_{g-1}, c_{g-1}, d_{g-1}, a_{g-1}', c_{g-1}', d_{g-1}')$$

The image $\mathcal{F}_g = \mathcal{F}_g(T_g)$ is called the Fricke space of the closed Riemann surfaces of genus g. The topology of $\mathcal{F}_g$ is induced by the relative topology of $\mathcal{F}_g$ in $\mathbb{R}^{6g-6}$.

$\mathcal{F}_g$ is simply connected (§5.2), hence homeomorphic to all of $\mathbb{R}^{6g-6}$.

$\mathcal{F}_g : T_g \longrightarrow \mathcal{F}_g(T_g)$ is a bijective map. (Mia's GP).

Thus, a topology is induced on $T(R) \cong T_g$ from $\mathcal{F}_g$. Similarly, indentifying $\mathbb{R}^{6g-6} = \mathbb{C}^{3g-3}$ induces a complex structure.

We now turn to the more general version, following [BG].

Let p, n be fixed non-negative integers and define the numbers

$$a = 2p - 2 + n$$
$$d = 3p - 3 + n$$

We always assume that $a > 0$, so that $d \geq 0$.

Let X denote a topological oriented surface of topological type (p,n). Recall, this means that X is homeomorphic to a closed Riemann surface of genus p with n distinct points removed.

We are interested in orientation preserving homeomorphisms of X onto Riemann surfaces.

Two such homeomorphisms, $f_1 : X \to S_1$ and $f_2 : X \to S_2$, will be called equivalent if there is a conformal map $h : S_1 \to S_2$ such that the map

$$f_2^{-1} \circ h \circ f_1$$

is homotopic (or, equivalently here, isotopic) to the identity map.

The equivalence class of a map $f : X \to f(X)$ is denoted $[f]$.

The set of all equivalence classes is the Fricke space $\mathcal{F}(X)$.

Note: This is exactly our original definition of Teichmüller space.

We proceed to topologize this space.

The condition $a>0$ ensures that a Riemann surface of type (p,n) carries a Poincaré (hyperbolic) metric — unique complete Riemannian metric of Gaussian curvature (-1) consistent with the conformal structure on S.

Let C be a closed (non-oriented) curve on S.
We denote by $l_S(C)$ the infimum of Poincaré (hyperbolic) lengths of all closed curves on S freely homotopic to C.

This number depends only on the free homotopy class of C — if $l_S(C)$ is positive, then it is the length of the unique closed (Poincaré) geodesic freely homotopic to C.

Now, one sees that the number
$$l_{f(X)}(f(C)),$$
where C is now a closed curve on X, depends only on the equivalence class $[f]$ and the free homotopy class $[C]$.
We write
$$l_{[f]}(C) := l_{f(X)}(f(C)).$$

For a fixed C, $l_{[f]}(C)$ is a function from the Fricke space $\mathfrak{F}(X)$ to the set $\mathbb{R}_+ \cup \{0\}$ of non-negative reals.

We give $\mathfrak{F}(X)$ the weakest topology that makes all of these functions continuous.

Now, let Y be another surface of type (p,n).
Every homeomorphism $w: X \to Y$ induces a bijection w_*, called an allowable mapping, of $\mathfrak{F}(X)$ onto $\mathfrak{F}(Y)$. It is defined by

$$w_*[f] = [f \circ w^{-1}]$$

(RE) One easily verifies that this definition is legitimate, w_* depends only on the homotopy class of w, $id_* = id$, and $(w' \circ w)_* = (w')_* \circ (w)_*$ — where $w': Y \to Z$ is a homeo of Y to some surface Z of type (p,n).
Also, every allowable mapping w_* is a homeomorphism of $\mathfrak{F}(X)$ onto $\mathfrak{F}(w(X))$.

The allowable self-mappings of $\mathfrak{F}(X)$ form a group called the Fricke modular group of X, denoted $\mathfrak{F}\mathcal{M}(X)$.

An allowable mapping $w: X \to Y$, $w_*: \mathfrak{F}(X) \to \mathfrak{F}(Y)$, conjugates $\mathfrak{F}\mathcal{M}(X)$ onto $\mathfrak{F}\mathcal{M}(Y)$.

Two points $[f_1],[f_2] \in \mathcal{F}(X)$ are equivalent under $\mathcal{FM}(X)$ if and only if the Riemann surfaces $f_1(X)$ and $f_2(X)$ are conformally equivalent.

i.e., the Fricke modular group is the mapping class group!

All of this combines to tell us that the Fricke space and the Fricke modular group actually only depend on the topological type (p,n) of X.

Thus we have $\mathcal{F}_{p,n}$ and $\mathcal{FM}_{p,n}$.

Remarks. [BG] have essentially just renamed the Teichmüller space the Fricke space. The one truly new thing here is the construction of the topology. We could call this the Fricke topology, and regard the topologized Teichmüller space as Fricke space. So we will.