We begin this week by reconsidering Teichmüller space in terms of Beltrami coefficients.

Recall that for a marked Riemann surface $[S,f] \in \mathcal{T}(R)$, regarding the marking as an orientation preserving diffeo $f: R \to S$, the Beltrami coefficient is

$$\mu_f = \mu \frac{d\bar{z}}{dz}$$

where $\mu(z) = \dfrac{\partial_{\bar{z}} F}{\partial_z F}$ and $F$ is the local representation of $f$.

Let $B(R)_1$ be the space of all Beltrami coefficients on $R$. Define a topology on $B(R)_1$ using the $L^\infty$-norm,

$$\| \mu_f \|_\infty = \sup_{z \in R} |\mu_f(z)|$$

Let $\mathrm{Diff}_+(R)$ denote the space of orientation preserving diffeos of $R$ to itself. And $\mathrm{Diff}_0(R)$ the connected component of the identity in $\mathrm{Diff}_+(R)$.

$\mathrm{Diff}_0(R) \triangleleft \mathrm{Diff}_+(R)$.

These groups act on $B(R)_1$ by pullback

$$\tau: \mathrm{Diff}_+(R) \times B(R)_1 \to B(R)_1 \; : (\omega, \mu_f) \to \omega^*(\mu_f) = \mu_{f \circ \omega^{-1}} = \left( \frac{\overline{\partial_z \omega}}{\partial_z \omega} \, \frac{\mu_f - \mu_\omega}{1 - \overline{\mu_\omega} \mu_f} \right) \circ \omega^{-1}$$

Theorem [IT] 1.6

For orientation-preserving diffeos $f: R \to S$ and $g: R \to S'$, There exists a biholomorphic mapping $h: S \to S'$ if and only if the relation $\mu_g = w^*(\mu_f)$ holds for some $w \in \mathrm{Diff}_+(R)$.
Furthermore, $g^{-1} \circ h \circ f$ is homotopic to the identity of $R$ if and only if $w \in \mathrm{Diff}_0(R)$.

Proof. Suppose there exists a biholomorphic mapping $h: S \to S'$.
Setting $w = g^{-1} \circ h \circ f$, we have

$$\mu_g = \mu_{h \circ f \circ w^{-1}} = \mu_{f \circ w^{-1}} = w^*(\mu_f).$$

Conversely, if $w \in \mathrm{Diff}_+(R)$ with $\mu_g = w^*(\mu_f)$, then the chain rule formula for Beltrami coefficients (Prop 1.5) shows that $h := g \circ w \circ f^{-1}: S \to S'$ is biholomorphic. $\square$

Thus, we have,

Corollary. The mapping sending $(S, f) \in \mathcal{T}(R)$ to $\mu_f \in B(R)_1$ induces the following

$$\mathcal{T}(R) \cong B(R)_1 / \mathrm{Diff}_0(R)$$
$$\mathcal{M}_g \cong B(R)_1 / \mathrm{Diff}_+(R).$$

$\square$

We now reconstruct Teichmüller space from yet another point of view — conformal structures induced by Riemannian metrics on R.

Let $ds^2 = g = \langle , \rangle$ be a Riemannian metric on a real 2-dimensional oriented smooth manifold M. The metric can be represented as

$$ds^2 = E\,dx^2 + 2F\,dx\,dy + G\,dy^2 \qquad (*)$$

in a local coordinate nbd $(U, (x,y))$. Setting $z = x + iy$, we can write

$$ds^2 = \lambda \left| dz + \mu\, d\bar{z} \right|^2 \qquad (**)$$

where $\lambda$ is a positive smooth function on $U$ and $\mu$ is a complex-valued smooth function with $|\mu| < 1$ on $U$.

CBS —
$$\lambda = \frac{1}{4}\left(E + G + 2\sqrt{EG - F^2}\right)$$
and
$$\mu = \frac{E - G + 2iF}{E + G + 2\sqrt{EG - F^2}}$$

Local coordinates $(u,v)$ on $U$ are said to be isothermal coordinates for $ds^2$ if $ds^2$ is represented as

$$ds^2 = \rho\,(du^2 + dv^2)$$

on $U$ where $\rho$ is a positive smooth function on $U$.

Here the orientations of $(x,y)$ and $(u,v)$ coincide w/ the one on M.

The complex coordinate $w = u + iv$ is also called an isothermal coordinate for $ds^2$.

An isothermal coordinate $w$ for $ds^2$ satisfies

$$\rho |dw|^2 = \rho |\partial_z w|^2 \left| dz + \frac{\partial_{\bar z} w}{\partial_z w} d\bar z \right|^2.$$

Comparing this w/ (**), we see that an isothermal coordinate $w$ for $ds^2$ exists if the PDE

$$\frac{\partial w}{\partial \bar z} = \mu \frac{\partial w}{\partial z} \qquad (***)$$

has a diffeomorphic solution $w$. This is the Beltrami Equation:

CBS (Ch4, §2) such a solution $w$ always exists provided $\|\mu\|_\infty < 1$.

Hence, for an atlas $\{(U_\alpha, (x,y)_\alpha)\}$ of $M$, there exists an isothermal coordinate $w_\alpha$ on each $U_\alpha$.

Now $\{(U_\alpha, w_\alpha)\}$ defines a complex structure on $M$ (RE?).

Let $R$ denote the Riemann surface obtained in this way.

The complex structure on $R$ is called the conformal structure induced by $ds^2$.

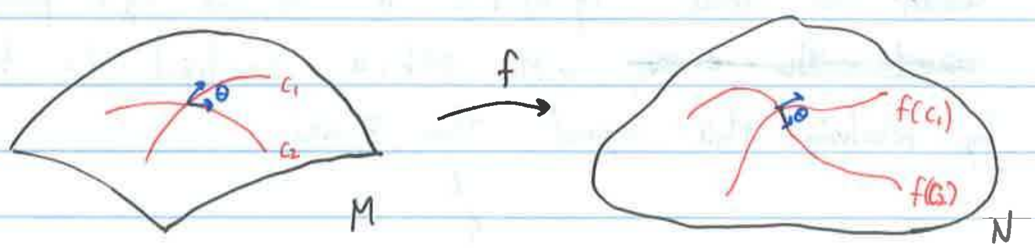
For oriented 2-dim Riemannian mflds $(M, ds_1^2)$ and $(N, ds_2^2)$ or $(M, g_M)$, $(N, g_N)$, an $f \in \mathrm{Diff}_+(M,N)$ is conformal if and only if there exists $\varphi \in \mathcal{F}(M)$ such that

$$f^*(g_N) = e^{\varphi} g_M.$$

where $f^*$ is the pullback along $f$.

Recall, $(f^* g_N)(x,y) = g_N(f_* x, f_* y)$ where $f_* : TM \to TN$ is the induced tangent map.

As in the cases we've already studied, conformal means that angles are preserved.




$\theta$ is supposed to be the same (I can't draw ")

$(M, g_M)$ and $(N, g_N)$ are said to be conformally equivalent iff their exists a conformal map between them.

The conformal structures induced by $g_M$ and $g_N$ are "the same".

The uniqueness of the representation (**) yields

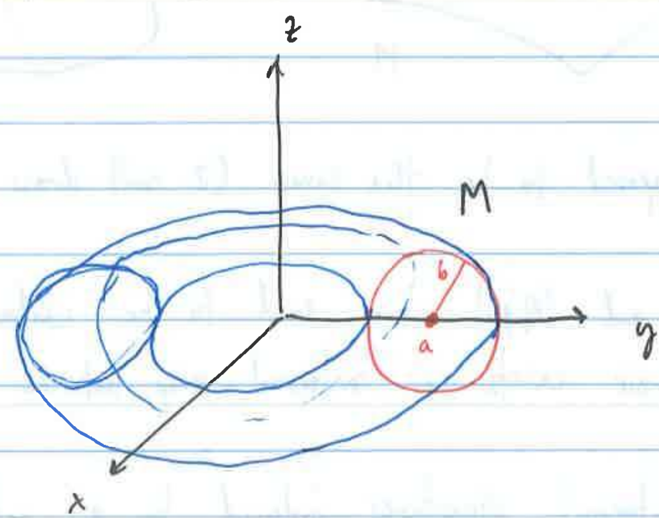
Theorem 1.7

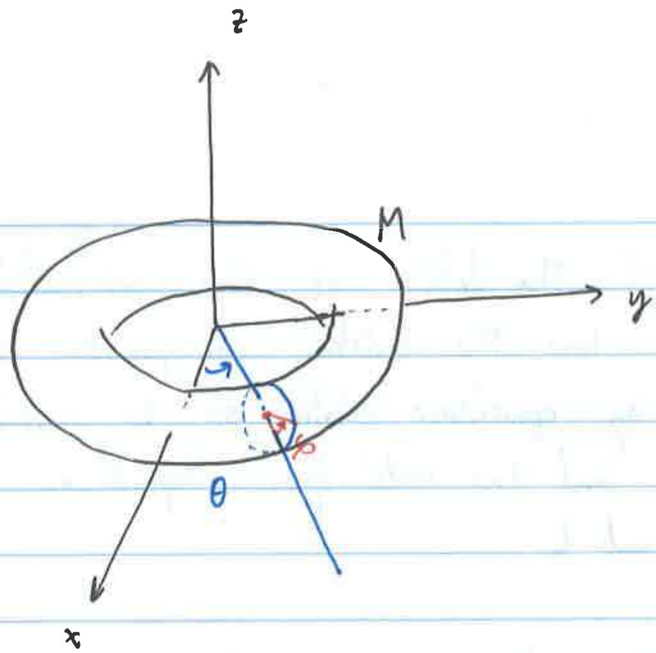
Let $R$ and $S$ be Riemann surfaces induced by oriented 2-dim Riemannian manifolds $(M, g_M)$ and $(N, g_N)$, respectively. Then $f: (M, g_M) \to (N, g_N)$ is conformal if and only if $f: R \to S$ is biholomorphic.

* Here "conformal" refers to the geometric def'n, not the complex analytic one, but the two notions coincide.

Example. The torus embedded in $\mathbb{R}^3$.

Consider the circle $(y-a)^2 + z^2 = b^2$ on the $(y,z)$-plane ~~around the z-axis~~, with $0 < b < a$. Construct the torus by revolving this around the $z$-axis






$$x = (a + b\cos\varphi)\cos\theta$$
$$y = (a + b\cos\varphi)\sin\theta$$
$$z = b\sin\varphi$$

M comes equipped w/ a natural metric $ds^2$ induced by the Euclidean metric of $\mathbb{R}^3$.

Local coordinates $(\theta, \varphi)$ can be induced from polar coordinates on $\mathbb{R}^3$ as above. Define

$$\psi := \psi(\varphi) = \int_0^{\varphi} \frac{b}{a + b\cos\varphi}\, d\varphi .$$

Then the metric $ds^2 = (a + b\cos\varphi)^2 d\theta^2 + b^2 d\varphi^2$ takes the form

$$ds^2 = \lambda(\psi)\left(d\theta^2 + d\psi^2\right).$$

Thus $(\theta, \psi)$ or $w = \theta + i\psi$ is an isothermal coordinate for $ds^2$ on M, which defines a complex structure on M. Thus the induced R is a torus.

"A little more calculation" shows that R is biholomorphic to $\mathbb{C}/\Gamma$, where $\Gamma$ is a lattice in $\mathbb{C}$ generated by 1 and $\frac{ib}{\sqrt{a^2 - b^2}}$.

$\boxed{\text{GP}}$ Work out all of the details of this construction and explain/show how this matches our previous construction of the torus by equivalence relation on $\mathbb{C}$. Explain how Teichmüller space (and the moduli space) depend on the choice of $a$ and $b$.

Now we construct Teichmüller space by Riemannian metrics.

Fix a closed Riemann surface of genus $g \geq 1$, and take any local coordinate $z$ on $R$.

For any metric $ds^2 = \lambda\, |dz + \mu\, d\bar{z}|^2$, we obtain a globally defined Beltrami coefficient $\mu$ on $R$, of type $(-1,1)$ and w/ $\|\mu\|_\infty < 1$.

This $\mu$ is called the Beltrami coefficient induced by the metric $ds^2$.

We wish to show that the space of Beltrami coefficients induced by metrics coincides w/ the space of those induced by orientation-preserving diffeos.

First, suppose $\mu$ is induced by a metric, so that we obtain a new Riemann surface w/ isothermal coordinates $\{(u,w)_\alpha\}$ — by solving the Beltrami equation.

Let $R'$ denote the Riemann surface w/ this new complex structure. Then the identity map $id : R \to R'$ is an orientation-preserving diffeo. and its Beltrami coefficient coincides w/ $\mu$.

Conversely, let $f \in \mathrm{Diff}_+(R,S)$. Take a Riemannian metric $g_S$ on $S$ so that the induced complex structure coincides with that of $S$.
In fact, by the Uniformization Thm, the universal cover of $S$ is either $\mathbb{C}$ (genus 1) or $\mathbb{H}$ (genus $\geq 2$). Take $\tilde{g}_S$ to be the flat or hyperbolic metric, then define $g_S$ to be the Riemannian metric induced by the covering.

Now the pull back $f^*(g_S)$ gives the same Beltrami coefficient as $f$.

<u>Note</u>: This does not depend on the choice of metric $g_S$. Such a metric $f^*(g_S) =: g_f$ is said to be a Riemannian metric on $R$ corresponding to $f$.

Thus the set $B(R)_1$ is equal to the set of Beltrami coefficients induced by Riemannian metrics.

Let $\mathrm{Riem}(R)$ be the set of Riemannian metrics on $R$.
We have

$$\mathcal{T}(R) \cong \mathrm{Riem}(R) / \mathrm{Diff}_0(R)$$

$$\mathcal{M}_g \cong \mathrm{Riem}(R) / \mathrm{Diff}_+(R)$$

Here we say that $g_1$ and $g_2$ in $\mathrm{Riem}(R)$ are (conformally) equivalent iff there exists a map $f \in \mathrm{Diff}_+(R)$ that is conformal, and strongly (or Teichmüller) equivalent iff there is a map $f \in \mathrm{Diff}_0(R)$ that is conformal.

Obviously, conformal means $f^* g_2 = e^{\varphi} g_1$ in this context.

<u>Next</u>: We attempt to study Fricke spaces.