[IT] §1.4.2

We now investigate the geometric meaning of Beltrami coefficients.

Let D be a domain containing $0 \in \mathbb{C}$ in the complex z-plane and D' a domain in the complex w-plane. Let f be an orientation-preserving diffeomorphism of D onto D'.

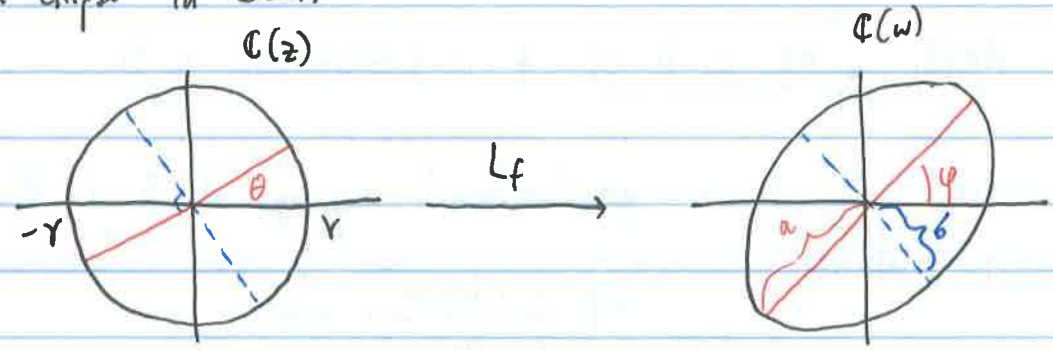
Let $L_f(z) = \partial_z f(0)\, z + \partial_{\bar z} f(0)\, \bar z$ be the first-order Taylor approximation of f at $0 \in D$.

Since $f \in \mathrm{Diff}_+$, its Jacobian is positive, thus

$$Df(0) = |\partial_z f(0)|^2 - |\partial_{\bar z} f(0)|^2 > 0$$

In particular, this implies that $\partial_z f(0) \neq 0$ and $|\mu(0)| = \left|\frac{\partial_{\bar z} f(0)}{\partial_z f(0)}\right| < 1$.

Moreover, the linear map L_f sends a circle centered at $0 \in \mathbb{C}(z)$ to an ellipse in $\mathbb{C}(w)$.



$\theta = \frac{1}{2} \arg \mu(0)$

$\varphi = \theta + \arg \partial_z f(0)$

$a = (1 + |\mu(0)|)\, r\, |\partial_z f(0)|$

$b = (1 - |\mu(0)|)\, r\, |\partial_z f(0)|$

By the inequalities

$$|\partial_z f(0)|\,(1-|\mu(0)|)\,|z| \leq |L_f(0)| \leq |\partial_z f(0)|\,(1+|\mu(0)|)\,|z|$$

the ratio of the major axis to the minor axis (the eccentricity) of this ellipse is

$$K(0) = \frac{1+|\mu(0)|}{1-|\mu(0)|}.$$

This statement holds at every point in D. Or equivalently, this holds in every local coordinate chart on R.

Thus, the Beltrami coefficient

$$\mu_f(z) = \frac{\partial_{\bar{z}} f(z)}{\partial_z f(z)} \quad z \in D$$

is said to be the complex dilation of f at z.

Recall, $\mu_f = 0$ on D iff f is biholomorphic on D.

Defn. We call f a quasiconformal mapping of D to D' iff f satisfies

$$K_f = \sup_{z \in D} \frac{1+|\mu_f(z)|}{1-|\mu_f(z)|} < \infty.$$

* Note: This K is not curvature!

Such an f is said to be quasiconformal w/ Beltrami coefficient μ_f. K_f is called the maximal dilation of f.

The change-of-charts formula for μ_f implies that $|\mu_f|$ (absolute value) of an orientation-preserving diffeomorphism $f: R \to S$ does not depend on local coordinates. Thus, $|\mu_f|$ is continuous on R w/ $|\mu_f| < 1$.

Since R is compact, we get

$$\|\mu_f\|_\infty := \sup_{z \in R} |\mu_f(z)| < 1. \qquad (*)$$

In particular,

$$K_f = \sup_{z \in R} \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|} = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty} < \infty. \qquad (**)$$

Hence, any orientation-preserving diffeomorphism $f: R \to S$ is a quasiconformal mapping of R.

Prop 1.5. For Riemann surfaces R, S, T, and orientation-preserving diffeomorphisms $f: R \to S$, $g: S \to T$,

$$\mu_g \circ f = \frac{\partial_z f}{\overline{\partial_z f}} \, \frac{\mu_{g\circ f} - \mu_f}{1 - \overline{\mu_f} \cdot \mu_{g \circ f}}. \qquad (***)$$

In particular, for $f_i \in \mathrm{Diff}_+(R, S_i)$ $i = 1, 2$, the mapping $f_2 \circ f_1^{-1}: S_1 \to S_2$ is biholomorphic if and only if $\mu_{f_1} = \mu_{f_2}$.

Proof. RE! (Apply the Chain Rule.)

Remark. Formula (***) has the same form as a biholomorphic automorphism

$$\gamma(z) = e^{i\theta} \frac{z-\alpha}{1-\bar{\alpha}z}$$

of the unit disc $\mathbb{D}$, $\theta \in \mathbb{R}$, $\alpha \in \mathbb{D}$.

Moreover, (***) shows that for a fixed f, the Beltrami coefficient $\mu_{g\circ f}$ depends holomorphically on μ_g.

Next week – Spaces of Beltrami Coefficients
Construction of Teichmüller space via Riemannian metrics
Fricke space