Riemann Surfaces
Lecture 3

We begin today by considering the fundamental groups of the spheres and torus.

In groups of peers, discuss the following examples. Feel free to ask me any questions, but please discuss your questions with your group first.

Example 1 S^n is simply connected for $n \geq 2$.

Example 2 S^1 is *not* simply connected. Compute the fundamental group of S^1 as follows.

Let $\pi : \mathbb{R} \to S^1 : t \mapsto (\cos(2\pi t), \sin(2\pi t))$ map the real line onto S^1 (considered as the unit circle in $\mathbb{R}^2$). For $n \in \mathbb{Z}$ let $L_n : [0,1] \to \mathbb{R} : s \mapsto n \cdot s$. This map, called left-translation by n, maps the interval $[0,1]$ to the interval $[0,n]$. Now define a map $\varphi_n : [0,1] \to S^1$ by $\varphi_n = \pi \circ L_n$.

Let $p = (1,0) \in S^1$, and define a function $\Phi : \mathbb{Z} \to \pi_1(S^1, p)$ by $\Phi(n) = [\varphi_n]$, the homotopy class of φ_n. Recall that since S^1 is connected, $\pi_1(S^1, p) = \pi_1(S^1)$.

Goal: Show that Φ is an isomorphism of groups.

This can be done by proving the following statements.

Lemma 3 Let $\alpha : [0,1] \to \mathbb{R}$ be a path such that $\alpha(0) = 0$ and $\alpha(1) = n$. Then $[\pi \circ \alpha] = [\varphi_n] \in \pi_1(S^1)$.

Corollary 4 If $\beta : [0,1] \to \mathbb{R}$ with $\beta(0), \beta(1) \in \mathbb{Z}$ and $\beta(1) - \beta(0) = n \in \mathbb{Z}$, then $[\pi \circ \beta] = [\varphi_n] \in \pi_1(S^1)$.

Proposition 5 Φ is a homomorphism; *i.e.*, Φ preserves group operations.

Remark 6 The proofs of the *-ed lemmas below (7 and 9) are a bit involved. At this moment, you should consider them as "well known" and use them to prove the propositions that follow. We'll actually prove versions of them each together in class later this week or early next week.

Lemma 7* Let $f : [0,1] \to S^1$ be a path in S^1 with $f(0) = f(1) = p$. Then there exists a unique *lift* $\widetilde{f} : [0,1] \to \mathbb{R}$ such that $\pi \circ \widetilde{f} = f$ and $\widetilde{f}(0) = 0$. Note: since $\pi(\widetilde{f})(1) = p$, then $\widetilde{f}(1) \in \mathbb{Z}$.

Use Lemma 7* to prove:

Proposition 8 Φ is an epimorphism (surjective).

Lemma 9* Suppose $G : [0,1] \times [0,1] \to S^1$ ~~be~~ is a homotopy of loops at p between a loop $f(s) = F(s,0)$ and the constant loop $p(s) \equiv p$. Then there exists a map $\widetilde{F} : [0,1] \times [0,1] \to \mathbb{R}$ such that $\pi \circ \widetilde{F} = F$.

Remark 10 The previous lemma is an example of "Homotopy Lifting."

Use Lemma 9* to prove:

Proposition 11 Φ is a monomorphism (injective).

Taking the three propositions together, you have proved

Theorem 12 $\pi_1(S^1) = \mathbb{Z}$.

This concludes the example. ◇

Remark 13 Consider $S^1 \hookrightarrow \mathbb{C}$ as the unit circle. The loops φ_n above may be realized as $\varphi_n(t) = e^{2\pi i n t}$ for $t \in [0,1]$. Then the homotopy classes of loops $[\varphi_n]$ in $\pi_1(S^1)$ essentially count the number of times that each loop "winds around" the origin $0 \in \mathbb{C}$. The connection with the winding number of complex analysis should be obvious to those familiar with it.

The result of the previous example can be used to justify the following claim about the fundamental group of the flat torus.

Example 14 $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$.

We now turn our attention to

Covering Spaces

Definition 15 Local homeomorphisms.

Let $\widetilde{M}, M$ be manifolds. A map $\pi : \widetilde{M} \to M$ is said to be a local homeomorphism iff each pt $p \in \widetilde{M}$ has a neighborhood $U \ni p$ such that $\pi(U)$ is open in M and $\pi|_U$ is a homeomorphism.

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Definition 16 Coverings.

A local homeomorphism $\pi : \widetilde{M} \to M$ is called a covering (map) iff $\forall p \in M$ has a connected neighborhood $V \ni p$ s.t. every connected component of $\pi^{-1}(V)$ is mapped homeomorphically onto V.

Then $\widetilde{M}$ is called a covering (space) of M.

If M is smooth and $\pi : \tilde{M} \to M$ is a local homeo. then $\tilde{M}$ can be made smooth via pullback: Define an atlas on $\tilde{M}$ by

$$\begin{tikzcd} V_\beta \arrow[d, hook] & U_\alpha \arrow[d, hook] \\ \tilde{M} \arrow[r, "\pi"] \arrow[d, "\psi_\beta"'] & M \arrow[d, "\varphi_\alpha"] \\ \mathbb{R}^n \arrow[r, "\varphi_\alpha \circ \pi \circ \psi_\beta^{-1}"'] & \mathbb{R}^n \end{tikzcd}$$

w/ $\pi|V_\beta$ a homeo and $\varphi_\alpha \circ \pi \circ \psi_\beta^{-1}$ smooth.

π is then called a local diffeomorphism.

Lemma 17 *If $\pi : \widetilde{M} \to M$ is a covering, then each point is covered the same number of times. That is, $\pi^{-1}(p)$ has the same number of elements for each $p \in M$.*

Proof Recommended Exercise.

Definition 18 Lifts.

Let $\pi : \widetilde{M} \to M$ be a covering and $f : N \to M$ a continuous map. A continuous map $\tilde{f} : N \to \widetilde{M}$ with $\pi \circ \tilde{f} = f$ is called a lift of f to $\widetilde{M}$.

The diagram commutes:

$$\begin{tikzcd} & \widetilde{M} \arrow[d, "\pi"] \\ N \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] \arrow[r, phantom, shift left=3, "\circ"] & M \end{tikzcd}$$

Lemma 19 *Let $\pi : \widetilde{M} \to M$ be a covering, $p \in M$, $q \in \pi^{-1}(p)$, and $\gamma : [0,1] \to M$ a curve with $\gamma(0) = p$. Then γ can be lifted to a curve $\widetilde{\gamma} : [0,1] \to \widetilde{M}$ with $\widetilde{\gamma}(0) = q$, so that $\pi \circ \widetilde{\gamma} = \gamma$. Moreover, $\widetilde{\gamma}$ is uniquely determined by the choice of $q \in \widetilde{M}$.*

Proof

Let $T := \{ t \in [0,1] \mid \gamma|_{[0,t]}$ can be lifted to a unique $\tilde{\gamma}|_{[0,t]}$ w/ $\tilde{\gamma}(0) = q \}$.

Clearly $0 \in T$, so $T \neq \emptyset$.

If $t \in T$, choose a neighborhood V of $\gamma(t)$ s.t. $\pi^{-1}(V)$ is homeo to V for each connected component.

Let $\tilde{V}$ denote the conn. component of $\pi^{-1}(V)$ containing $\tilde{\gamma}(t)$.

Choose $\varepsilon > 0$ s.t. $\gamma([t, t+\varepsilon]) \subset V$.

Then $\tilde{\gamma}$ can be uniquely extended as a lift of γ to $[0, t+\varepsilon]$, since $\pi|_{\tilde{V}}$ is a homeo.

Thus T is open in $[0,1]$.

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Now suppose $(t_n) \subset T$ w/ $(t_n) \to t_0 \in [0,1]$.

Choose a neighborhood V of $\gamma(t_0)$ as before.

Then, $\exists\ n_0 \in \mathbb{N}$ w/ $\gamma([t_{n_0}, t_0]) \subset V$.

Let $\tilde{V}$ be the conn. component of $\pi^{-1}(V)$ containing $\tilde{\gamma}(t_{n_0})$.

Again, $\tilde{\gamma}$ can be extended to $[0, t_0]$.

Hence $t_0 \in T$ and T is closed.

Thus $T = [0,1]$.

$\square$

Lemma 20 Let $\pi : \widetilde{M} \to M$ be a covering, and $\Gamma : [0,1] \times [0,1] \to M$ a homotopy of paths $\gamma_0 := \Gamma(\cdot, 0)$ and $\gamma_1 = \Gamma(\cdot, 1)$ with fixed endpoints $\gamma_0(0) = \gamma_1(0) = p_0$ and $\gamma_0(1) = \gamma_1(1) = p_1$. Let $q_0 \in \pi^{-1}(p_0)$. The Γ can be lifted to a homotopy $\widetilde{\Gamma} : [0,1] \times [0,1] \to \widetilde{M}$ with initial point q_0, so that $\pi \circ \widetilde{\Gamma} = \Gamma$. In particular, the lifted paths $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ have the same initial and terminal points, and are homotopic in $\widetilde{M}$.

Proof

By the previous lemma, $\Gamma(\cdot, s)$ can be lifted to a path $\tilde{\gamma}_s$ w/ initial point q_0.

Put $\tilde{\Gamma}(t,s) := \tilde{\gamma}_s(t)$.

We must show $\tilde{\Gamma}$ is continuous.

Let $\Sigma := \{(t,s) \in [0,1] \times [0,1] \mid \tilde{\Gamma} \text{ is continuous at } (t,s)\}$.

Let $\tilde{U}$ be an open nbd of q_0 such that $\pi : \tilde{U} \to U$ is a homeomorphism onto a nbd U of p_0. Let $\varphi = (\pi|_{\tilde{U}})^{-1}$.

Since $\Gamma(\{0\} \times [0,1]) = p_0$ and Γ is continuous, there exists $\varepsilon > 0$ s.t. $\Gamma([0,\varepsilon] \times [0,1]) \subset \tilde{U}$.

By Lemma 19, $\tilde{\gamma}_s|_{[0,\varepsilon]} = \varphi \circ \tilde{\gamma}_s|_{[0,\varepsilon]}$ for all $s \in [0,1]$.

Hence $\tilde{\Gamma} = \varphi \circ \Gamma$ on $[0,\varepsilon] \times [0,1]$.

In particular, $(0,0) \in \Sigma$.

Now let $(t_0, s_0) \in \Sigma$. Choose a nbd $\tilde{\mathcal{U}}$ of $\tilde{\Gamma}(t_0, s_0)$ for which $\pi : \tilde{\mathcal{U}} \to \mathcal{U}$ is a homeo. onto a nbd $\mathcal{U}$ of $\Gamma(t_0, s_0)$. Again put $\varphi = (\pi|_{\tilde{\mathcal{U}}})^{-1}$.

Since $\tilde{\Gamma}$ is continuous at (t_0, s_0), we have $\tilde{\Gamma}(t, s) \in \tilde{\mathcal{U}}$ for $|t - t_0| < \varepsilon$, $|s - s_0| < \varepsilon$, $\varepsilon > 0$ "small enough".

Again, by uniqueness of lifting,

$$\tilde{\gamma}_s(t) = \varphi \circ \gamma_s(t) \quad \text{for} \quad |t - t_0|, |s - s_0| < \varepsilon.$$

Thus, $\tilde{\Gamma} = \varphi \circ \Gamma$ on $\{|t - t_0| < \varepsilon\} \times \{|s - s_0| < \varepsilon\}$.

$\tilde{\Gamma}$ continuous on a nbd of $(t_0, s_0) \Rightarrow \Sigma$ is open.

<u>RE</u>. Prove that Σ is closed.

Hence $\Sigma = [0,1] \times [0,1]$.

Now, since $\Gamma(\{1\} \times [0,1]) = p_1$ and $\pi \circ \tilde{\Gamma} = \Gamma$, then $\tilde{\Gamma}(\{1\} \times [0,1]) \subset \pi^{-1}(p_1)$. But $\pi^{-1}(p_1)$ is discrete since π is a covering and $\tilde{\Gamma}(\{1\} \times [0,1])$ is connected. Hence $\tilde{\Gamma}(\{1\} \times [0,1])$ must be a point.

Thus, all curves $\tilde{\gamma}_s$ share the same endpoint. $\square$

$$\begin{tikzcd} & \widetilde{M} \arrow[d, "\pi"] \\ N \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & M \end{tikzcd}$$

These lemmas can be used to prove

Theorem 21 *Let $\pi : \widetilde{M} \to M$ be a covering, N a simply connected manifold, and $f : N \to M$ a continuous map. Then there exists a continuous map $\widetilde{f} : N \to \widetilde{M}$ with $\pi \circ \widetilde{f} = f$.*

Remark 21 This is a generalization of Lemma 7*.

Proof

Choose $y_0 \in N$ and put $p_0 = f(y_0)$.

Choose $q_0 \in \pi^{-1}(p_0) \subset \widetilde{M}$.

For any $y \in N$, choose a path $\gamma : [0,1] \to N$ w/ $\gamma(0) = y_0$ and $\gamma(1) = y$.

By Lemma 19, the path $g := f \circ \gamma$ can be lifted to a path $\tilde{g}$ starting at q_0.

Put $\tilde{f}(y) := \tilde{g}(1)$.

Since N is simply connected, any two paths γ_1 and γ_2 w/ $\gamma_1(0) = \gamma_2(0) = y_0$ and $\gamma_1(1) = \gamma_2(1) = y$ are homotopic.

Since f is continuous, $g_1 = f(\gamma_1)$ and $g_2 = f(\gamma_2)$ are also homotopic.

Lemma 20 $\Rightarrow$ the point $\tilde{f}(y)$ is independent of path γ.

<u>RE</u>. Prove $\tilde{f}$ is continuous.

<u>Corollary</u>. Let $\pi : \widetilde{M} \to M$ be a covering, $\gamma : [0,1] \to M$ a loop w/ $\gamma(0) = \gamma(1) = p$, and $\tilde{\gamma} : [0,1] \to \widetilde{M}$ a lift of γ. If γ is null-homotopic, then $\tilde{\gamma}$ is closed and null homotopic. $\square$