Riemann Surfaces – Summer 2016

Part I.1 – Topological Foundations

Def'n. Let X be a set of points. A topology on X is a family τ of distinguished subsets, called open sets, fulfilling
1.) The union of any collection of open sets is open, and
2.) The intersection of any finite collection of open sets is open.

Note: $\emptyset$ and X are in τ

Def'n. A subset $S \subset X$ is said to be closed if its complement $X \setminus S$ is open.

Note: $\emptyset$ and X are also both closed.

Def'n. A topological space (X, τ) is said to be Hausdorff if and only if given any two points p_1 and p_2, $p_1 \neq p_2$, there exist open sets U_1 and U_2 such that $p_1 \in U_1$, $p_2 \in U_2$, and $U_1 \cap U_2 = \emptyset$.

Def'n. Let $f: X \to Y$ be a map of topological spaces. The preimage of a point $p \in Y$ is the set

$$f^{-1}(p) = \{q \in X \mid f(q) = p\}.$$

The preimage of a set $U \subset Y$ is

$$f^{-1}(U) = \{q \in X \mid f(q) \in U\}.$$

Defn. A map $f: X \to Y$ of topological spaces is said to be continuous if and only if for every open set $V \subset Y$, the preimage $f^{-1}(V) \subset X$ is open.

Defn. A map $f: X \to Y$ of topological spaces is an isomorphism if it is both mono (injective, "1-1") and epi (surjective, "onto"). A continuous isomorphism $f: X \to Y$ for which $f^{-1}: Y \to X$ is also continuous is called a homeomorphism.

Defn. A topological space X is called locally Euclidean if and only if for each point $p \in X$ there exists an open set $U \ni p$ and a homeomorphism $\varphi: U \to V \subset \mathbb{R}^n$ where $V = \varphi(U)$ is an open subset of $\mathbb{R}^n$.

The pair (U, φ) is called a chart on X (centered) at p (if $\varphi(p) = 0$).

Defn. A topological space X is said to be disconnected if it is the union of two disjoint nonempty open sets. Otherwise, X is connected.

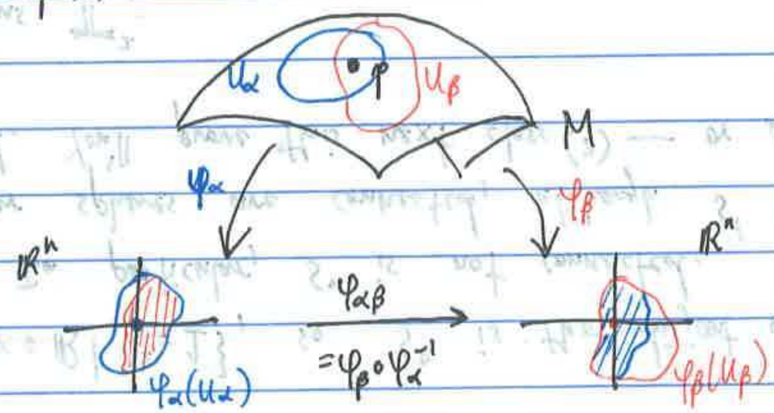
Defn. (According to Jost): A manifold is a connected locally Euclidean Hausdorff space, M.

The dimension of M is the dimension n of the open sets $\varphi(U) \in \mathbb{R}^n$ to which open sets in M are locally homeomorphic.

Def'n. An atlas on M is a family of charts $\{(U_\alpha, \varphi_\alpha)\}$ such that $\{U_\alpha\}$ constitute an open cover of M. An atlas is called smooth if all transition functions

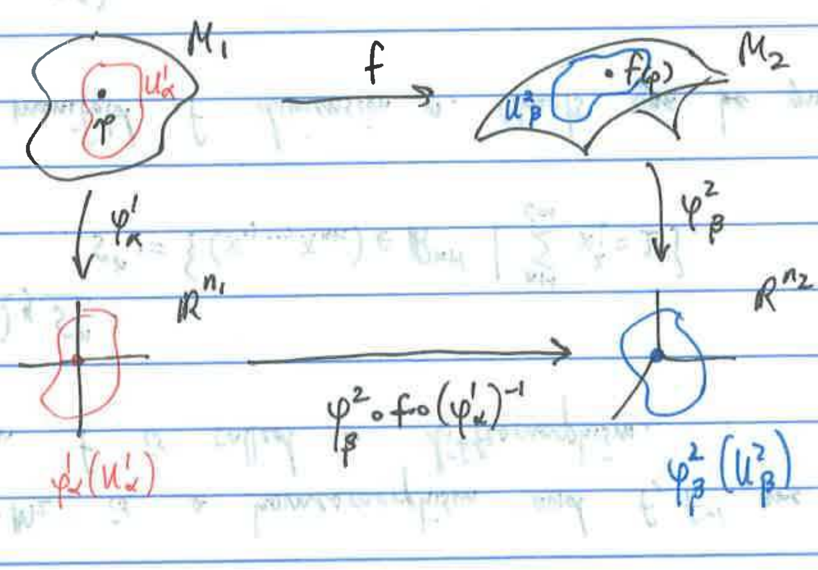
$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

are smooth (of class C^∞), whenever they are defined (i.e., when $U_\alpha \cap U_\beta \neq \emptyset$).



These transition functions form a cocycle.

Def'n. Let M_1 and M_2 be smooth manifolds and $f: M_1 \to M_2$ a continuous map map between them. We say that f is smooth if all local representations $\varphi^2_\beta \circ f \circ (\varphi^1_\alpha)^{-1} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ are smooth.



⊛ Define a Differential structure on M!

Def'n. If $f: M_1 \to M_2$ is a homeomorphism and f, f^{-1} are both smooth, then f is called a diffeomorphism.

Example. The Sphere(s), S^n:

$$S^n := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}.$$

This is a manifold of dimension n. Charts can be given as follows:

$U_1 := S^n \setminus (0, \dots, 0, 1)$

$\varphi_1(x_1, \dots, x_{n+1}) := \left(\frac{x_1}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}}\right).$

$U_2 := S^n \setminus \{(0, \dots, 0, -1)\}$

$\varphi_2(x_1, \dots, x_{n+1}) := \left(\frac{x_1}{1+x_{n+1}}, \dots, \frac{x_n}{1+x_{n+1}}\right)$

RE. Verify that the transition function $\varphi_{12}: \mathbb{R}^n \to \mathbb{R}^n$ is smooth.

Notes: $S^0 = \{x \in \mathbb{R} \mid x^2 = 1\}$, so S^0 is the disjoint union of two points. In particular, S^0 is not connected.
All other spheres are connected, although S^1 is not simply connected. You'll prove this next class (?) — or soon.

Example. The torus, $\mathbb{T}^2$:
Let $\mathbb{C}_0$ denote the complex plane with the origin removed, the "punctured plane" $\mathbb{C} \setminus \{0\}$.
Let $w_1, w_2 \in \mathbb{C}_0$, such that $\frac{w_1}{w_2} \notin \mathbb{R}$. That is, the vectors $w_1, w_2 \in \mathbb{R}^2$ are linearly independent.
Define an equivalence relation on $\mathbb{C}$ by declaring $z_1 \sim z_2$ iff there exist $n, m \in \mathbb{Z}$ such that

$$z_1 - z_2 = n w_1 + m w_2.$$

RE. Verify that this is an equivalence relation.

Let π be the projection mapping $z \in \mathbb{C}$ to its equivalence class $[z]_\sim$. The torus is $\mathbb{T}^2 := \pi(\mathbb{C})$.

$\mathbb{T}^2$ can be made a manifold as follows. Let $\Delta_\alpha \subset \mathbb{C}$ be an open set for which no two points are equivalent, then set

$$U_\alpha := \pi(\Delta_\alpha) \quad \text{and}$$

$$\varphi_\alpha := (\pi|_{\Delta_\alpha})^{-1}$$

RE. Find a "minimal" atlas for $\mathbb{T}^2$.

RE. Verify that the transition functions are smooth. (Justify)

Example. $\mathbb{R}^n$, $\mathbb{C}^n$, and all open subsets of each are smooth manifolds that are not compact.

Note: $\mathbb{T}^2$ and S^n are compact (for all n).