Name:____________________________

Math 829F: Riemann Surfaces
Midterm Exam

**Instructions**: Answer all questions, showing as much work as necessary to prove that you understand your answer. You may use a single $8\frac{1}{2} \times 11$ in$^2$ two-sided page of your own hand-written notes.

**1 – 5.** [6 points each] True or False? Write **T** if the statement is always true or **F** otherwise in the space provided next to the question number. If you write **F**, use the space beneath the statement to justify your answer.

_____**1.** The fundamental group of the sphere is $\pi_1(S^2) = \{0\}$.

_____**2.** Let $w_1, w_2 \in \mathbb{C}$ with $\frac{w_2}{w_1} \notin \mathbb{R}$, and $m, n \in \mathbb{Z}^+$ (in particular, $m, n \neq 0$). The torus $\mathrm{T}^2_{m,n}$ generated by $mw_1$ and $nw_2$ is a universal covering of the torus $\mathrm{T}^2$ generated by $w_1$ and $w_2$.

_____**3.** The torus $\mathrm{T}^2$ can be endowed with a flat conformal metric.

_____**4.** The Riemann sphere $S^2$ can be endowed with a flat conformal metric.

_____**5.** Every geodesic on the torus $\mathrm{T}^2$ is closed.

Questions **6** through **12** are worth 10 points each.

**6.** Consider the statement: *For a connected manifold $M$ and distinct points $p, q \in M$, $\pi_1(M, p) \cong \pi_1(M, q)$.* Describe the isomorphism used in the proof. (You don't need to prove it's an isomorphism.)

**7.** Let $I^2 \subset \mathbb{R}^2$ be a fundamental domain for the torus $\mathrm{T}^2$, with projection map $\pi : I^2 \to \mathrm{T}^2$. Give an example of two curves in $I^2$ whose images in $\mathrm{T}^2$ are *not* in the same homotopy class.

**8.** Let $S$ be a surface endowed with a Riemannian metric. Prove that the exponential map $\exp_p : T_pS \to S$ is a local diffeomorphism from a neighborhood of $0 \in T_pS$ to a neighborhood of $p \in S$. [Hint: Compute the derivative at $0 \in T_pS$.]

**9.** Let $w_1, w_2 \in \mathbb{C}$ with $\frac{w_2}{w_1} \notin \mathbb{R}$. Let $\mathrm{T}^2$ be the torus constructed by identifying $z \sim z + mw_1 + nw_2$, and $C$ the cylinder constructed by identifying $z \sim z + kw_1$ for $z \in \mathbb{C}$, $m, n, k \in \mathbb{Z}$. As constructed, $\pi : C \to T^2$ is a covering. What is the corresponding group of covering transformations, $H_\pi$?

**10.** Show that a non-constant holomorphic map $f : \mathrm{T}^2 \to \mathrm{T}^2$ between tori must be unramified.

**11.** Let $\Gamma < PSL(2, \mathbb{R})$ be a discrete group of isometries acting freely and properly discontinuously on $(H, g_H)$, such that the quotient $\Gamma \setminus H =: S$ is compact. Give an example of a closed geodesic in $S$. What is its length?

**12.** Let $\Gamma < PSL(2, \mathbb{R})$ be generated by a single hyperbolic transformation of $H$. Suppose that the generator $\gamma$ fixes 0 and $P \neq \infty$ in $\partial H$. Sketch and describe a Dirichlet fundamental polygon for the action of $\Gamma$.