Name: Key

Math 829F: Riemann Surfaces
Midterm Exam

Instructions: Answer all questions, showing as much work as necessary to prove that you understand your answer. You may use a single $8\frac{1}{2} \times 11$ in^2 two-sided page of your own hand-written notes.

1 – 5. [6 points each] True or False? Write **T** if the statement is always true or **F** otherwise in the space provided next to the question number. If you write **F**, use the space beneath the statement to justify your answer.

T 1. The fundamental group of the sphere is $\pi_1(S^2) = \{0\}$.

S^2 is simply connected.

F 2. Let $w_1, w_2 \in \mathbb{C}$ with $\frac{w_2}{w_1} \notin \mathbb{R}$, and $m, n \in \mathbb{Z}^+$ (in particular, $m, n \neq 0$). The torus $\mathrm{T}^2_{m,n}$ generated by mw_1 and nw_2 is a universal covering of the torus T^2 generated by w_1 and w_2.

$T^2_{m,n}$ is a covering, but it cannot be universal since it is not simply connected.

T 3. The torus T^2 can be endowed with a flat conformal metric.

The Euclidean metric on $\mathbb{C}$ is H_π-invariant, hence defines a flat metric on T^2.

F 4. The Riemann sphere S^2 can be endowed with a flat conformal metric.

$\chi(S^2) = 2$. Thus, by Gauss-Bonnet, any metric on S^2 must satisfy $\int_{S^2} K_\lambda \, dA_\lambda = 2\pi\chi(S^2) = 4\pi$. But a flat metric would have $K_\lambda \equiv 0$.

F 5. Every geodesic on the torus T^2 is closed.

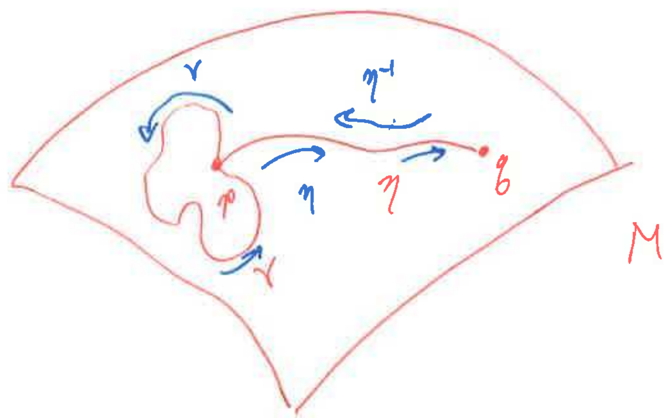
Any line ℓ in $\mathbb{C}$ w/ irrational slope projects to a geodesic in T^2 that is not closed.

Questions **6** through **12** are worth 10 points each.

6. Consider the statement: *For a connected manifold M and distinct points $p, q \in M$, $\pi_1(M,p) \cong \pi_1(M,q)$.* Describe the isomorphism used in the proof. (You don't need to prove it's an isomorphism.)

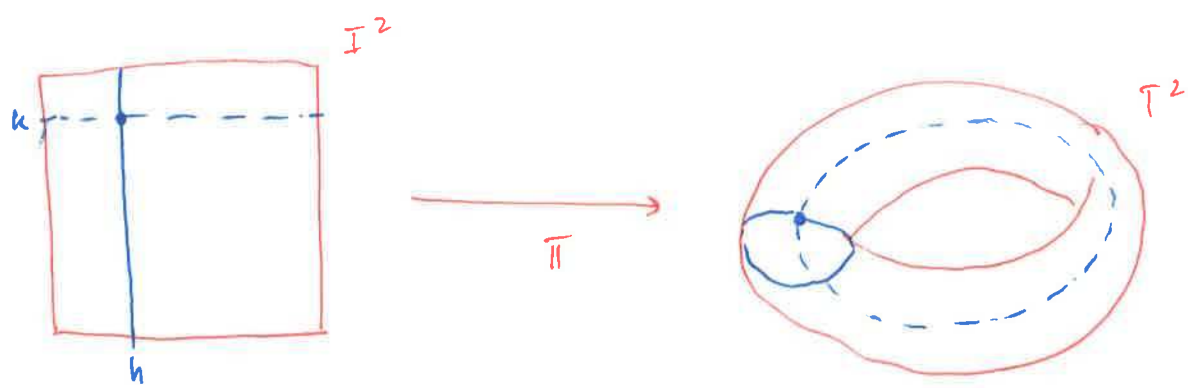
Choose a path η in M connecting p and q. Then for any $\gamma \in \pi_1(M,p)$, the path $\chi_\eta(\gamma) = \eta\gamma\eta^{-1}$ is an element of $\pi_1(M,q)$.

This map, $\chi_\eta : \pi_1(M,p) \to \pi_1(M,q)$ is an isomorphism of fundamental groups. In general, it depends on the choice of η.



7. Let $I^2 \subset \mathbb{R}^2$ be a fundamental domain for the torus T^2, with projection map $\pi : I^2 \to \mathrm{T}^2$. Give an example of two curves in I^2 whose images in T^2 are *not* in the same homotopy class.

Let $h, k \in (0,1)$. The lines $x = h$ and $y = k$ project to curves in T^2 that are not homotopic.



8. Let S be a surface endowed with a Riemannian metric. Prove that the exponential map $\exp_p : T_pS \to S$ is a local diffeomorphism from a neighborhood of $0 \in T_pS$ to a neighborhood of $p \in S$. [Hint: Compute the derivative at $0 \in T_pS$.]

By definition, $\exp_p(v) = \gamma_{p,v}(1)$ where $\gamma_{p,v}$ is the unique geodesic through $p \in S$ with initial velocity $\dot\gamma_{p,v}(0) = 1$. We compute

$$\begin{aligned} \exp_{p*0}(v) &= \frac{d}{dt} \exp_p(tv)\Big|_{t=0} \\ &= \frac{d}{dt} \gamma_{p,tv}(1)\Big|_{t=0} \\ &= \frac{d}{dt} \gamma_{p,v}(t)\Big|_{t=0} \\ &= \dot\gamma_{p,v}(0) \\ &= v. \end{aligned}$$

Thus, $\exp_{p*0} = Id$ is the identity map, and the Inverse Function Theorem applies. The InFT then immediately yields the result.

9. Let $w_1, w_2 \in \mathbb{C}$ with $\frac{w_2}{w_1} \notin \mathbb{R}$. Let T^2 be the torus constructed by identifying $z \sim z + mw_1 + nw_2$, and C the cylinder constructed by identifying $z \sim z + kw_1$ for $z \in \mathbb{C}$, $m, n, k \in \mathbb{Z}$. As constructed, $\pi : C \to T^2$ is a covering. What is the corresponding group of covering transformations, H_π?

The group $G_\pi = \{(k,0) \mid k \in \mathbb{Z}\} \cong \mathbb{Z}$ is a subgroup of the fundamental group $\pi_1(T^2) \cong \mathbb{Z}^2$.

The group of covering transformations is thus

$$\begin{aligned} H_\pi &= \pi_1(T^2)/G_\pi \\ &= \mathbb{Z}^2/\mathbb{Z} \end{aligned}$$

$$\boxed{H_\pi = \mathbb{Z}.}$$

10. Show that a non-constant holomorphic map $f : \mathrm{T}^2 \to \mathrm{T}^2$ between tori must be unramified.

By Riemann-Hurwitz, $\chi(T^2) = m\,\chi(T^2) - \nu_f$ where m is the degree of f and ν_f is the total ramification order. But $\chi(T^2) = 0$ implies

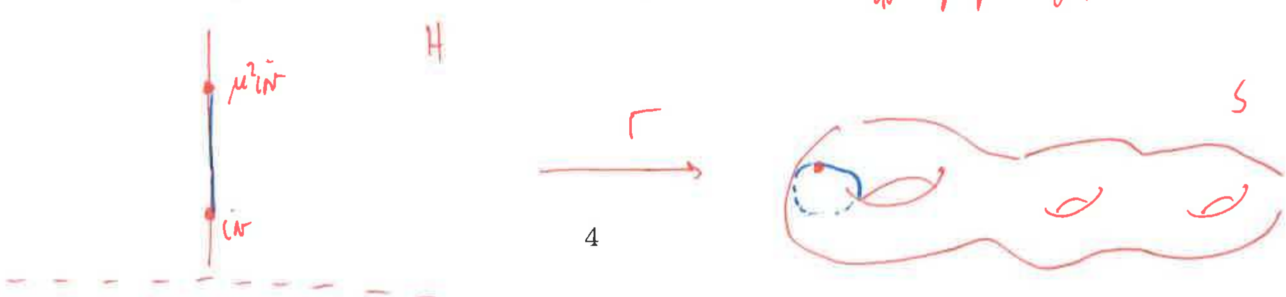
$$0 = m \cdot 0 - \nu_f$$
$$= -\nu_f.$$

Thus, the total ramification of f must be 0. $\square$

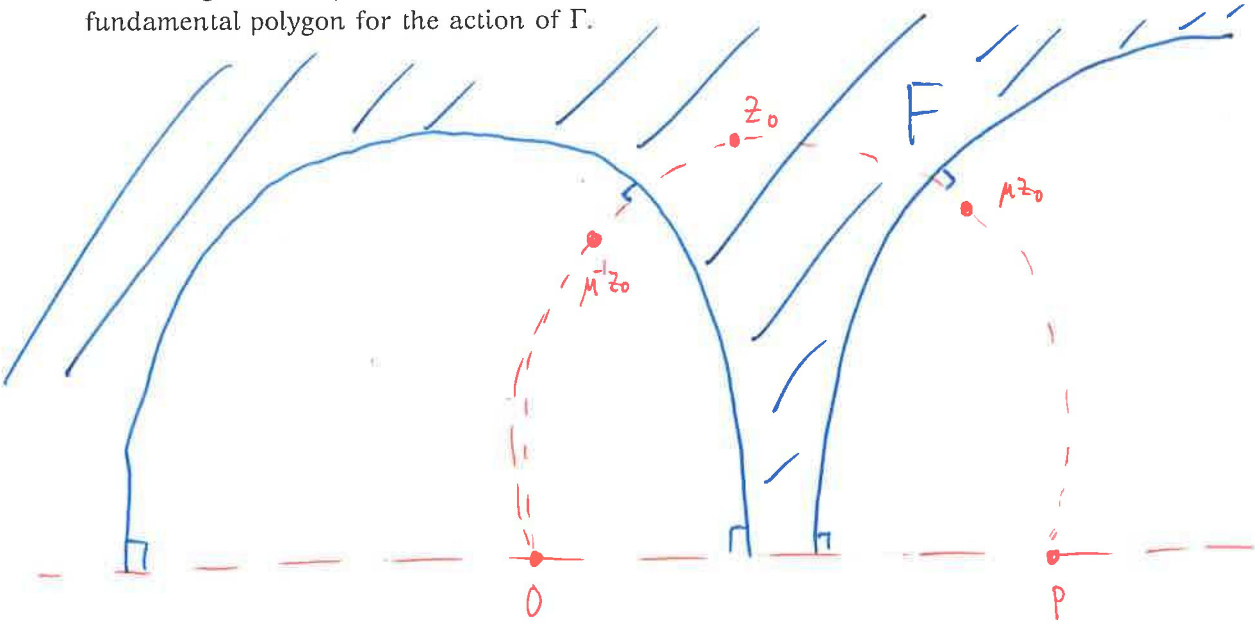
11. Let $\Gamma < PSL(2,\mathbb{R})$ be a discrete group of isometries acting freely and properly discontinuously on (H, g_H), such that the quotient $\Gamma \setminus H =: S$ is compact. Give an example of a closed geodesic in S. What is its length?

WLOG, assume $\gamma \in \Gamma$ fixes 0 and ∞ in ∂H. Then the imaginary axis $i\mathbb{R}$ is the only geodesic in H left invariant under the action of γ.

If $\gamma(z) = \mu^2(z)$, $\mu \neq 1$, then γ identifies the points iv and $\mu^2 iv$ in $i\mathbb{R}$. The segment of $i\mathbb{R}$ connecting these points is a closed geodesic in S. Its length is $l = \int_{iv}^{\mu^2 iv} \frac{1}{y}\,dy = \log(\mu^2)$.



12. Let $\Gamma < PSL(2, \mathbb{R})$ be generated by a single hyperbolic transformation of H. Suppose that the generator γ fixes 0 and $P \neq \infty$ in ∂H. Sketch and describe a Dirichlet fundamental polygon for the action of Γ.



The points in H equivalent to z_0 lie on the circle through z_0, 0, and P. The geodesics that form the boundary of F are circles orthogonal to $\partial H = \mathbb{R}$ for which each pt is equidistant (in hyperbolic distance) to z_0 and μz_0 (or resp. $\mu^{-1} z_0$).