Riemann Surfaces
Homework 2 solutions

1. Let A be an $\mathbb{R}$-algebra, and let $\mathrm{Der}(A)$ denote the space of derivations of A. Show that (a) if $D \in \mathrm{Der}(A)$ and $k \in \mathbb{R}$, then $D(k) = 0$; and (b) if $D_1, D_2 \in \mathrm{Der}(A)$, then their *commutator* $[D_1, D_2] = D_1D_2 - D_2D_1$ is also in $\mathrm{Der}(A)$.

Solution (a) Let $a \in A$. On one hand, $D(ka) = kD(a)$ by linearity. On the other hand, $D(ka) = D(k)a + kD(a)$ by the defining property of a derivation. Thus $kD(a) = D(k)a + kD(a)$ for all $a \in A$ implies that $D(k) = 0$.

(b) Let $a, b \in A$ and compute

$$\begin{aligned}
[D_1, D_2]\,(ab) &= D_1(D_2(ab)) - D_2(D_1(ab)) \\
&= D_1(D_2(a)b + aD_2(b)) - D_2(D_1(a)b + aD_1(b)) \\
&= D_1(D_2(a)) \cdot b + D_2(a) \cdot D_1(b) + D_1(a) \cdot D_2(b) + a \cdot D_1(D_2(b)) \\
&\quad - (D_2(D_1(a)) \cdot b + D_1(a) \cdot D_2(b) + D_2(a) \cdot D_1(b) + a \cdot D_2(D_1(b))) \\
&= D_1(D_2(a)) \cdot b + a \cdot D_1(D_2(b)) - D_2(D_1(a)) \cdot b - a \cdot D_2(D_1(b)) \\
&= [D_1(D_2(a)) - D_2(D_1(a))] \cdot b + a \cdot [D_1(D_2(b)) - D_2(D_1(b))] \\
&= [D_1, D_2]\,(a) \cdot b + a \cdot [D_1, D_2]\,(b).
\end{aligned}$$

Thus $[D_1, D_2]$ satisfies the defining property of a derivation. That $[D_1, D_2]$ is linear follows easily from the linearity of D_1 and D_2. □

2. Let $g_\lambda = \lambda^2(z)dz\,d\bar{z}$ be a conformal Riemannian metric on a Riemann surface S. Show that the angle between two vectors $x, y \in T_pS$ as measured with respect to g_λ is exactly the angle between x and y determined by the dot product on T_pS regarded as $\mathbb{R}^2$. How are the distances $d_{E,p}$ and $d_{\lambda,p}$ related? (Here $d_{E,p}$ is the Euclidean distance $d(x, y)_p := \sqrt{(x - y) \cdot (x - y)}$ in the vector space T_pS, and $d_{\lambda,p}$ is defined analogously with respect to the inner product $g_{\lambda,p}$.)

Solution We first consider the distance $d_\lambda(x, y)_p$ between tangent vectors

$x, y \in T_P S$. We have

$$\begin{aligned} d_\lambda(x,y)_p &= \|x-y\|_{\lambda,p} \\ &= \sqrt{g_\lambda(x-y,x-y)_p} \\ &= \sqrt{\lambda^2(p)\,dz\,d\bar{z}(x-y,x-y)} \\ &= \sqrt{\lambda^2(p)\,\|x-y\|_E^2} \\ &= \lambda(p)\,\|x-y\|_{E,p} \\ &= \lambda(p) d_E(x,y)_p. \end{aligned}$$

Now the angle between x and y in T_pS is given by

$$\begin{aligned} \theta_\lambda(x,y)_p &= \frac{g_\lambda(x,y)_p}{\|x\|_{\lambda,p}\,\|y\|_{\lambda,p}} \\ &= \frac{\lambda^2(p) g_E(x,y)_p}{\lambda(p)\,\|x\|_{E,p}\;\lambda(p)\,\|y\|_{E,p}} \\ &= \frac{g_E(x,y)_p}{\|x\|_{E,p}\,\|y\|_{E,p}} \\ &= \theta_E(x,y)_p. \end{aligned}$$

Thus a conformal Riemannian metric preserves the measure of angles, but not necessarily distances, with respect to the Euclidean metric. □

3. Consider the torus $\mathbb{T}^2$ as constructed in Chapter 1. Regard $\pi : \mathbb{C} \to \mathbb{T}^2$ as the universal covering with covering transformation group $H_\pi \cong \mathbb{Z}^2$. Suppose $g_\lambda = \lambda^2(z) dz\, d\bar{z}$ is a conformal Riemannian metric on $\mathbb{C}$ for which each $\varphi \in H_\pi$ is an isometry. Then g_λ defines a conformal Riemannian metric on $\mathbb{T}^2$. Show that the curvature $K_\lambda([z]) = -\Delta \log \lambda([z])$ satisfies

$$\int_{\mathbb{T}^2} K_\lambda = 0.$$

This justifies calling $\mathbb{T}^2$ (as constructed) a *flat* torus.

Solution Choose $w_1 = 1$ and $w_2 = i$ in the construction of $\mathbb{T}^2$, so that I^2 is a fundamental domain in $\mathbb{C}$; that is, for every $z \in \mathbb{C}$ there exists a point in $z_0 \in I^2$ such that $\pi(z) = \pi(z_0)$, whereas no two points in the interior of I^2 are in the same equivalence class.

Now recall the definitions of curvature, Laplacian, and the area element for a conformal Riemannian metric. In conformal local coordinates $z = x + iy$ regarded as coordinates in Euclidean $\mathbb{R}^2$, we have

$$\begin{aligned} K_\lambda &= -\Delta_\lambda \log(\lambda), \\ \Delta_\lambda &= \frac{1}{\lambda^2}\Delta_E, \\ dA_\lambda &= \frac{i}{2}\lambda^2\, dz \wedge d\bar{z} = \lambda^2\, dx \wedge dy = \lambda^2\, dA_E. \end{aligned}$$

In the last equality we usually suppress the wedge product and simply write $dA_E = dx \wedge dy = dx\,dy$, as in Calculus III.

Applying these definitions to the current situation, we have

$$\begin{aligned} \int_{\mathbb{T}^2} K_\lambda &= \int_{\mathbb{T}^2} K_\lambda\, dA_\lambda \\ &= \iint_{I^2} -\frac{1}{\lambda^2}\Delta_E \log(\lambda)\,\lambda^2\, dA_E \\ &= \iint_{I^2} \left(-\frac{\partial^2}{\partial x^2}\log(\lambda) - \frac{\partial^2}{\partial y^2}\log(\lambda)\right) dA_E \end{aligned}$$

in local coordinates on $I^2 \subset \mathbb{R}^2$. Now Green's theorem applies with $Q = -\frac{\partial}{\partial x}\log\lambda$ and $P = \frac{\partial}{\partial y}\log\lambda$, transforming the area integral to a path integral along the positively-oriented boundary,

$$\int_{\mathbb{T}^2} K_\lambda = \int_{\partial I^2} P\,dx + Q\,dy. \tag{1}$$

The assumption that g_λ, hence also λ, is H_π-invariant on $\mathbb{C}$ now implies that P and Q are H_π-invariant. In particular, the values of P at π-equivalent points along ∂I^2 coincide. The same is true for Q. Opposite sides are oppositely-oriented with respect to the path integral (1). Thus this integral is 0. □