Riemann Surfaces
Homework 1 solutions

1. Construct a 3-dimensional torus $\mathbb{T}^3$ by defining an appropriate equivalence relation on $\mathbb{R}^3$, following the construction of $\mathbb{T}^2$ in the book. Be sure to prove that the equivalence relation you define is indeed an equivalence relation, and to provide the charts. Give an example of what a transition function $\varphi_{\alpha\beta} : \mathbb{R}^3 \to \mathbb{R}^3$ would look like. You do not need to prove that the transition functions are smooth (although it should be obvious).

Solution Choose $w_1, w_2, w_3 \in \mathbb{R}^3$ linearly independent, and define a relation on $\mathbb{R}^3$ by $x_1 \sim x_2$ if and only if there exists $(m, n, k) \in \mathbb{Z}^3$ such that

$$x_2 - x_1 = mw_1 + nw_2 + kw_3.$$

For every $x \in \mathbb{R}^3$, $x - x = \mathbf{0} = 0w_1 + 0w_2 + 0w_3$, so $\sim$ is reflexive. Suppose $x_1 \sim x_2$ so that there exists $(m, n, k) \in \mathbb{Z}^3$ such that $x_2 - x_1 = mw_1 + nw_2 + kw_3$. Then $x_1 - x_2 = -mw_1 - nw_2 - kw_3$ with $(-m, -n, -k) \in \mathbb{Z}^3$ and $x_2 \sim x_1$. Thus $\sim$ is symmetric. Now suppose $x_1, x_2, x_3 \in \mathbb{R}^3$ with $x_1 \sim x_2$ and $x_2 \sim x_3$. Then there exist $(m_1, n_1, k_1), (m_2, n_2, k_2) \in \mathbb{Z}^3$ such that $x_2 - x_1 = m_1w_1 + n_1w_2 + k_1w_3$ and $x_3 - x_2 = m_2w_1 + n_2w_2 + k_2w_3$. Then

$$x_3 - x_1 = (m_1 + m_2)w_1 + (n_1 + n_2)w_2 + (k_1 + k_2)w_3$$

with $(m_1 + m_2, n_1 + n_2, k_1 + k_2) \in \mathbb{Z}^3$, so $x_1 \sim x_3$ and $\sim$ is transitive. Therefore $\sim$ is an equivalence relation.

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^3/\sim : x \mapsto [x]_\sim$ be the projection defined by $\sim$, and define $\pi(\mathbb{R}^3) = \mathrm{T}^3$. Define charts on T^3 as follows. Let Δ_α be a connected open set in $\mathbb{R}^3$ for which no two points are equivalent with respect to $\sim$ and define $U_\alpha = \pi(\Delta_\alpha)$. Then $\pi|_{\Delta_\alpha}$ is a homeomorphism; put $\varphi_\alpha := (\pi|_{\Delta_\alpha})^{-1}$. The pair $(U_\alpha, \varphi_\alpha)$ define a chart on T^3.

To understand the transition functions, consider two such charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$. Any point $x \in U_{\alpha\beta}$ has local coordinates

$$\begin{aligned} \varphi_\alpha(x) &= y + m_1w_1 + n_1w_2 + k_1w_3, \quad \text{and} \\ \varphi_\beta(x) &= y + m_2w_1 + n_2w_2 + k_2w_3 \end{aligned}$$

for some $y \in \mathbb{R}^3$ and $(m, n, k)_i \in \mathbb{Z}^3$, $i = 1, 2$. The transition function is then given by

$$\varphi_{\alpha\beta}(x) = x + (m_2 - m_1)w_1 + (n_2 - n_1)w_2 + (k_2 - k_1)w_3$$

for all $x \in \varphi_a(U_{\alpha\beta})$. This is just translation by the fixed vector $(m_2 - m_1)w_1 + (n_2 - n_1)w_2 + (k_2 - k_1)w_3$ in $\mathbb{R}^3$ and is clearly smooth. $\diamond$

2. Prove that the lift $\widetilde{f} : N \to \widetilde{M}$ of theorem 1.3.1 is continuous.

Solution We begin by recalling how $\widetilde{f}$ was defined. Choose an arbitrary point $y_0 \in N$. For $y \in N$, choose a path $\gamma : I \to N$ with $\gamma(0) = y_0$ and $\gamma(1) = y$. Now $g := f \circ \gamma : I \to M$ is a path which can be lifted to $\widetilde{M}$ by Lemma 1.3.2. Define $\widetilde{f}(y) = \widetilde{g}(1)$. By Lemma 1.3.3 and since N is simply connected, the value $\widetilde{f}(y)$ is independent of the choice of path γ.

Now let $\Sigma = \{y \in N \mid \widetilde{f} \text{ is continuous at } y\}$. Suppose $y \in \Sigma$ and choose a neighborhood V of $\widetilde{f}(y)$ such that $\pi|_V$ is a homeomorphism onto $U := \pi(V) \subset M$. Write $\varphi := (\pi|_V)^{-1}$. Now since f is continuous, there exists an open neighborhood $W \ni y$ in N such that $f(W) \subset U$, and $\widetilde{f}|_W = \varphi \circ f$. Since $\widetilde{f}$ is the composition of continuous functions, it is continuous on W. Therefore Σ is open.

Now let $(y_n) \subset \Sigma$ with $y_n \to y \in N$. Let U be a neighborhood of $f(y) \in M$ such that π^{-1} is a homeomorphism to each of its connected components. Since f is continuous, there exists a neighborhood W of y with $f(W) \subset U$, and an integer n_0 such that $y_n \in W$ for $n \geq n_0$. Now let V denote the connected component of $\pi^{-1}(f(W))$ containing $\widetilde{f}(y_{n_0})$ in $\widetilde{M}$, and write $\varphi = (\pi|_V)^{-1}$, as before. Again, we have that $\widetilde{f}|_W = \varphi \circ f$ is a composition of continuous functions. Thus $\widetilde{f}$ is continuous on W, and in particular at y. Therefore Σ is closed.

By Lemma 1.3.2, if $\gamma : I \to N$ is any path, then $(f \circ \gamma)$ can be lifted to $(\widetilde{f} \circ \gamma)$. Therefore Σ is nonempty and $\widetilde{f}$ is continuous on N. $\square$

3. Prove Corollary 1.3.1: *Let $\pi : \widetilde{M} \to M$ be a covering, and $\gamma : I \to M$ a null-homotopic loop in M. Then a lift $\widetilde{\gamma} : I \to \widetilde{M}$ is closed an null-homotopic in $\widetilde{M}$.*

Solution Let $\Gamma : I^2 \to M$ be the homotopy between $\gamma = \Gamma(\cdot, 0)$ and $p = \gamma(0) = \Gamma(\cdot, 1)$. By Lemma 1.3.3 (not 1.3.2!), this lifts to a homotopy $\widetilde{\Gamma} : I^2 \to \widetilde{M}$ between $\widetilde{\gamma} := \widetilde{\Gamma}(\cdot, 0)$ and $\widetilde{\Gamma}(\cdot, 1) \equiv q \in \pi^{-1}(p)$. $\square$