Selected Topics in College Geometry

With an Introduction to Non-Euclidean Geometries

Revised Edition

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Contents

Preface

This course is designed for a variety of students: those majoring or minoring in mathematics, secondary math-ed majors, and any student who wishes to further their knowledge of geometry beyond what they learned in high school. These notes have been written because I have not found a current book that I like for this course. I have written these notes to use until an acceptable book appears.

To the Student

The typical equipment for studying geometry are: pencil, paper, straightedge and compass. The study of geometry requires drawing many figures to illustrate and help understand the problem at hand, so a good compass and straightedge is essential. When reading these notes, the student should carefully go through each definition, theorem and proof to make sure they understand what is being said or done.

Working exercises is an essential requirement for the understanding of any course in mathematics. You should work as many exercises as possible beyond those assigned by the instructor. An exercise is not completed until the student goes back through the exercise and understands what was needed and used to obtain the solution. I call these the "tools of the trade." By working exercises, we build up a tool box of tools that can be used to solve problems we encounter later. Success in any mathematics course is measured by how full your tool box is.

Please let the author of these notes know of any errors you may find. I can be reached by email at either of the following addresses:
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Chapter 1

Euclid and Hilbert

1.1 Euclid - The Formal Beginning

The following material was taken from a translation of Euclid's Elements by Sir Thomas L. Heath. (*Euclid the Thirteen Books of The Elements Translated with Introduction and Commentary*, Vol 1, Second edition by Sir Thomas L. Heath, Dover Publications, 1956). This work (three volumes in all) not only contains a translation of Euclid but also a large amount of historical and mathematical commentary by Heath. We can see how Euclid formalized the treatment of geometry by starting with a set of definitions and postulates. From a modern viewpoint, his definitions certainly lack clarity.

BOOK I

DEFINITIONS

1. A point is that which has no part.
2. A line is breadthless length.
3. The extremities of a line are points.
4. A straight line is a line which lies evenly with the points on itself.
5. A surface is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A plane surface is a surface which lies evenly with the straight lines on itself.
8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie on a straight line.

9. And when the lines containing the angle are straight, the angle is called rectilinear.

10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular.

11. An obtuse angle is an angle greater than a right angle.

12. An acute angle is an angle less than a right angle.

13. A boundary is that which is an extremity of anything.

14. A figure is that which is contained by any boundary or boundaries.

15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;

16. And the point is called the center of the circle.

17. A diameter of a circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

18. A semicircle is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.

19. Rectilinear figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilaterals those contained by four, and multilateral those contained by more than four straight lines.

20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.

21. Further of trilateral figure, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.

22. Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right- angled. And let quadrilaterals other than these be called trapezia.

23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

POSTULATES

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

COMMON NOTIONS (Axioms)

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

BOOK I. PROPOSITIONS

Proposition 1.

On a given straight line to construct an equilateral triangle.

Let AB be the given finite straight line.

Thus it is required to construct an equilateral triangle on the straight line AB.

With center A and distance AB let the circle BCD be described; [Post. 3] again with center B and distance BA let circle ACE be described; [Post. 3] and from point C, in which the circles cut one another, to the points A, B let the straight lines CA, CB be joined. [Post. 1]

Now, since the point A is the center of the circle CDB, AC is equal to AB. [Def. 15]

Again, since the point B is the center of circle CAE, BC is equal to BA. [Def. 15]
But CA was also proved equal to AB; therefore each of the straight lines CA, CB is equal to AB.
And things which are equal to the same thing are also equal to one another; [C.N. 1]

therefore CA is also equal to CB.

Therefore the three straight lines CA, AB, BC are equal to one another.

Therefore the triangle ABC is equilateral; [Def. 20]
and it has been constructed on the given finite line AB

(Being) what it was required to do.

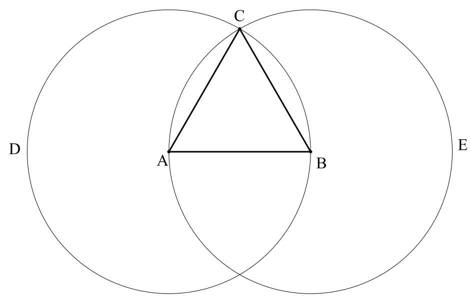


Figure 1.1.1: Proposition 1

Proposition 2.

To place at a given point (as an extremity) a straight line equal to a given straight line.

Let A be the given point, and let BC the given straight line. Thus it is required to place at the point A (as an extremity) a straight line equal to the given straight line BC

From the point A to the point B let the straight line AB be joined; [Post. 1]
and on it let the equilateral triangle DAB be constructed. [Prop. 1]

Let straight lines AE, BF be produced in a straight line with DA, DB; [Post. 2]
with center B and distance BC let the circle CGH be described; [Post. 3]
and again with center D and distance DG let circle GKL be described. [Post. 3]

Then, since the point B is the center of the circle CGH,

BC is equal to BG.

Again, since the point D is the center of the circle GKL

DL is equal to DG.

And in these DA is equal to DB; therefore the remainder AL is equal to the remainder BG. [C.N. 3]
But BC was proved to be equal to BG; therefore each of the straight lines AL, BC is equal to BG and things which are equal to the same thing are also equal to one another; [C.N. 1]
therefore AL is also equal to BC.

Therefore at the given point A the straight line AL is placed equal to the given straight line BC.

(Being) what it was required to do.

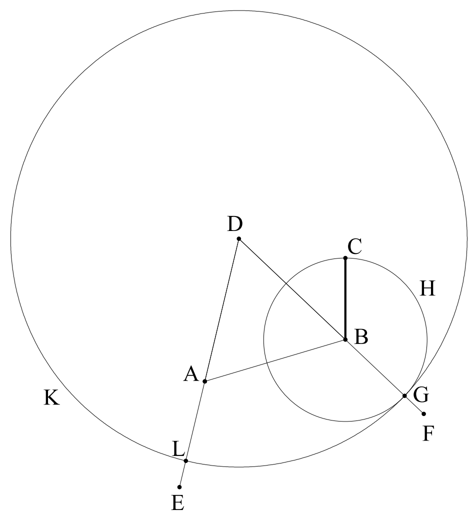


Figure 1.1.2: Proposition 2

Proposition 3.

Given two unequal straight lines, to cut off from the greater a straight line equal to the less.

Let AB, C be the two given unequal straight lines, and let AB be the greater of them.

Thus it is required to cut off from AB the greater a straight line equal to C the less.

At the point A let AD be placed equal to the straight line C; [Prop. 2]
and with center A and distance AD let the circle DEF be described. [Post. 3]

Now since the point A is the center of the circle DEF,

AE is equal to AD [Def. 15].

But C is also equal to AD. Therefore each of the straight lines AE, C is equal to AD; so that AE is also equal to C. [C.N. 1]

Therefore, given the two straight lines AB, C, from AB the greater AE has been cut off equal to C the less.

(Being) what it was required to do.

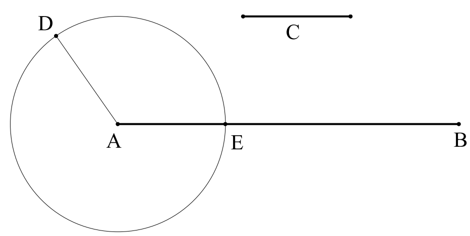


Figure 1.1.3: Proposition 3

The above figures are as they appeared in Euclid's work. His figures were merely illustrations for the propositions. Often much of the actual construction was not included; only the lines and circles important in the proof were drawn. On the next page is the complete construction for Proposition 3.

Exercises

1.1.1. *Given three coplanar points A, B, C. Use Proposition 2 to construct a circle with center at C and radius AB.*

1.1.2. *Repeat the construction in Figure 1.1.4 and explain the purpose and reason for all steps of the construction.*

1.1.3. *Assume you have only a collapsing compass. Given points A, B, C, D, with this compass only, construct the points of intersection of the circle with center C and radius CD and the line on AB.*

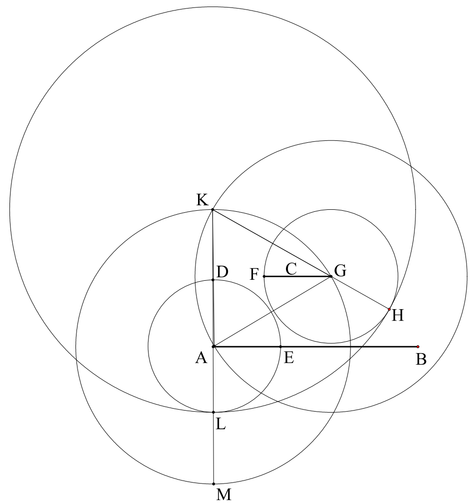


Figure 1.1.4: Proposition 3 - Complete Construction

Historical Note.

Euclid was born about 325 BCE and died about 265 BCE. Very little is known of his life except that he was a Greek mathematician who taught at Alexandria in Egypt and is best known for his work on geometry: *The Elements*. Actually, *The Elements* contained much more than just geometry; however, it is the geometry for which Euclid is best known. Even the portions of *The Elements* devoted to the theory of numbers is presented in a geometric setting. Euclid used the writings of many mathematicians before him and contemporary to him to create *The Elements*. He tried to formalize the mathematics known at the time. He had the advantage of working in Alexandria, which afforded him the vast resources of the Library at Alexandria. Euclid's work influenced the development of mathematics for more than 2000 years.

1.2 Hilbert - Revising Euclid

David Hilbert (1862-1943) was a German mathematician of great renown. Among his many contributions to mathematics was his version of the axioms of Euclidean geometry. A systematic study of these axioms led Hilbert to propose a revised set of axioms, or postulates. His book, *Grundlagen der Geometrie*, was published in 1899 and put geometry in a formal axiomatic setting. This book was a major influence in promoting the axiomatic approach to mathematics which has been one of the major characteristics of the subject throughout the 20th century.

Because of his reputation as being possibly the leading mathematician of the time, Hilbert was asked to address the Second International Congress of Mathematicians in Paris at the turn of the 20th century and give his prediction of the problems that would attract the attention of mathematicians in the 20th century. (This was probably a self-fulfilling prophecy, since if Hilbert considered a problem important, it must be important.) The resulting 23 soon-to-be-famous problems contained in his speech, *The Problems of Mathematics*, challenged (and continue to challenge) mathematicians to solve some very fundamental questions in mathematics.

Hilbert's Postulates for Plane Geometry[1]

PRIMITIVE TERMS

point, line, on, between, congruent

GROUP I: POSTULATES OF CONNECTION

I-1. *There is one and only one line passing through any two given distinct points.*

I-2. *Every line contains at least two distinct points, and for any given line there is at least one point not on the line.*

GROUP II: POSTULATES OF ORDER

II-1. *If point C is between points A and B, then A, B, C are all on the same line, and C is between B and A, and B is not between C and A, and A is not between C and B.*

II-2. *For any two distinct points A and B there is always a point C that is between A and B, and a point D that is such that B is between A and D.*

Definition 1.2.1. *By the* ***segment*** *AB is meant the points A and B and all points that are between A and B. Points A and B are called the end points of the segment. A point C is said to be on the segment AB if it is A or B or some point between A and B.*

[1]Material taken from A Survey of Geometry, Revised Edition, Howard Eves, Allyn and Bacon, 1972.

Definition 1.2.2. *Two lines, a line and a segment, or two segments, are said to* ***intersect*** *if there is a point that is on both of them.*

Definition 1.2.3. *Let A, B, C be three points not on the same line. Then by the* ***triangle*** *ABC is meant the three segments AB, BC, CA. The segments AB, BC, CA are called the* ***sides*** *of the triangle, and the points A, B, C are called the* ***vertices*** *of the triangle.*

II-3. (Pasch's Postulate) *A line that intersects one side of a triangle but does not pass through any of the vertices of the triangle must also intersect another side of the triangle.*

GROUP III: POSTULATES OF CONGRUENCE

III-1. *If A and B are distinct points and if A' is a point on a line m, then there are two and only two distinct points B' and B'' on m such that the pair of points A', B' is congruent to the pair A, B and the pair of points A', B'' is congruent to the pair A, B; moreover, A' is between B' and B''.*

III-2. *If two pairs of points are congruent to the same pair of points, then they are congruent to each other.*

III-3. *If point C is between points A and B and point C' is between points A' and B', and if the pair of points A, C is congruent to the pair A', C', and the pair of points C, B is congruent to the pair C', B', then the pair of points A, B is congruent to the pair A', B'.*

Definition 1.2.4. *Two segments are said to be* ***congruent*** *if the end points of the segments are congruent pairs of points.*

Definition 1.2.5. *By the* **ray** *AB is meant the set of all points consisting of those that are between A and B, the point B itself, and all points C such that B is between A and C. The ray AB is said to* ***emanate*** *from point A.*

Theorem 1.2.1. *If B' is any point on the ray AB, then the rays AB' and AB are identical.*

Definition 1.2.6. *By an* ***angle*** *is meant a point (called the* ***vertex*** *of the angle) and two rays (called the* ***sides*** *of the angle) emanating from the point. By virtue of the above theorem, if the vertex of the angle is point A and if B and C are any two points other than A on the two sides of the angle, we may unambiguously speak of the angle BAC (or CAB).*

Definition 1.2.7. *If ABC is a triangle, then the three angles BAC, CBA, ACB are called the* ***angles*** *of the triangle. Angle BAC is said to be* ***included*** *by the sides AB and AC of the triangle.*

III-4. *If BAC is an angle whose sides do not lie in the same line, and if A' and B' are two distinct points, then there are two and only two distinct rays, $A'C'$ and $A'C''$, such that angle $B'A'C'$ is congruent to angle BAC and* *angle $B'A'C''$ is congruent to angle BAC; moreover, if D' is any point on the ray $A'C'$ and D'' is any point on the ray $A'C''$*, then the segment $D'D''$ intersects the line determined by A' and B'.

III-5. *Every angle is congruent to itself.*

III-6. *If two sides and the included angle of one triangle are congruent, respectively, to two sides and the included angle of another triangle, then each of the remaining angles of the first triangle is congruent to the corresponding angle of the second triangle.*

GROUP IV: POSTULATE OF PARALLELS

IV-1. (Playfair's Postulate) *Through a given point A not on a given line m there passes at most one line that does not intersect m.*

GROUP V: POSTULATES OF CONTINUITY

V-1. (Postulate of Archimedes) *If A, B, C, D are four distinct points, then there is, on the ray AB, a finite set of distinct points $A_1, A_2, \ldots, A_n$ such that (1) each of the pairs $A, A_1;\ A_1, A_2;\ \ldots, A_{n-1}, A_n$ is congruent to the pair C, D, and (2) B is between A and A_n .*

V-2. (Postulate of Completeness) *The points of a line constitute a system of points such that no new points can be assigned to the line without causing the line to violate at least one of the eight postulates* I-l, I-2, II-l, II-2, 11-3, 111-1, III-2,V-1.

ALTERNATIVE GROUP V

Definition 1.2.8. *Consider a segment AB. Let us call one end point, say A, the **origin** of the segment, and the other point, B, the **extremity** of the segment. Given two distinct points M and N of AB, we say that M **precedes** N (or N **follows** M) if M coincides with the origin A or lies between A and N. A segment AB, considered in this way, is called an **ordered segment**.*

V′-1. (Dedekind's Postulate) *If the points of an ordered segment of origin A and extremity B are separated into two classes in such a way that*

(1) *each point of AB belongs to one and only one of the classes,*
(2) *the points A and B belong to different classes (which we shall respectively call the first class and the second class),*
(3) *each point of the first class precedes each point of the second class,*

then there exists a point C on AB such that every point of AB which precedes C belongs to the first class and every point of AB which follows C belongs to the second class.

1.3 Some Prerequisite Material

In this section we will review some definitions, theorems and other concepts that are normally talked about in a first course in geometry. Many of the theorems will be stated without proof. Our main goal is to introduce terminology and properties associated with points, lines, ratios, parallel lines, triangles, angles and circles.

Definition 1.3.1. *A set of points is said to be* **collinear** *if they all lie on the same line.*

Sometimes a particular set of collinear points is referred to as a **range of points**.

Definition 1.3.2. *A set of lines all passing through the same point are called* **concurrent lines** *and the common point they pass through is called the* **vertex**.

Another term that is used for a set of concurrent lines is a **pencil of lines**.

Definition 1.3.3. *A* **transversal** *is any line that cuts a collection of two or more lines.*

Definition 1.3.4. *A* **proportion** *is an expression of equality of two ratios.*

Theorem 1.3.1. *If* $\dfrac{a}{b} = \dfrac{c}{d}$, *then* $ad = bc$ *and the following hold:*

$$\begin{aligned} \frac{a}{c} &= \frac{b}{d} \\ \frac{d}{b} &= \frac{c}{a} \\ \frac{b}{a} &= \frac{d}{c} \\ \frac{a+b}{b} &= \frac{c+d}{d} \\ \frac{a-b}{b} &= \frac{c-d}{d} \end{aligned}$$

Theorem 1.3.2. *If* $\dfrac{a}{x} = \dfrac{b}{y},\ a \neq 0,\ b \neq 0$ *and* $a = b$, *then* $x = y$.

Theorem 1.3.3. *If* $\dfrac{a}{b} = \dfrac{c}{x}$ *and* $\dfrac{a}{b} = \dfrac{c}{y}$, *then* $x = y$.

Theorem 1.3.4. *If* $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \cdots$, *then* $\dfrac{a+c+e+\cdots}{b+d+f+\cdots} = \dfrac{a}{b}$.

Theorem 1.3.5. *If a line is parallel to one side of a triangle and intersects the other two sides, it divides these sides so that either side is to one of its segments as the other is to its corresponding segment.*

In triangle ABC, Figure 1.3.1, if DE is parallel to AB, then $\frac{CA}{CD} = \frac{CB}{CE}$ and $\frac{CA}{DA} = \frac{CB}{EB}$. Furthermore, from Theorem 1.3.6, we can also obtain $\frac{CD}{DA} = \frac{CE}{EB}$.

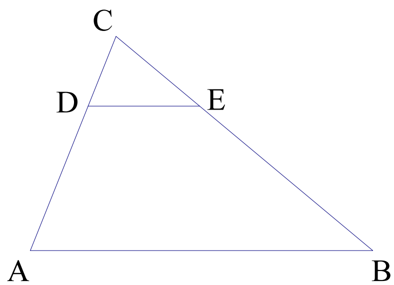


Figure 1.3.1

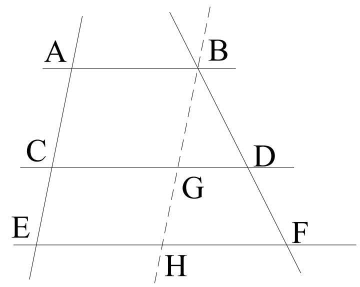


Figure 1.3.2

Theorem 1.3.6. *Parallel lines cut off proportional segments on two transversals.*

In Figure 1.3.2, we have by Theorem 1.3.6, $\frac{AC}{BD} = \frac{AE}{BF} = \frac{CE}{DF}$. To prove this result, draw $BH \parallel AE$ and apply Theorem 1.3.5.

Theorem 1.3.7. *If a line divides two sides of a triangle proportionally, then it is parallel to the third side.*

Theorem 1.3.8. *If a line divides two sides of a triangle so that either side is to one of its segments as the other side is to its corresponding segment, then the line is parallel to the third side.*

Definition 1.3.5. *An* **altitude** *of a triangle is the line through a vertex perpendicular to the side opposite the vertex.*

Definition 1.3.6. *The triangle formed by joining the feet of the altitudes of a given triangle is called the* **orthic triangle** *of the given triangle.*

Definition 1.3.7. *A* **median** *of a triangle is the line joining a vertex of a triangle to the midpoint of the opposite side.*

Definition 1.3.8. *A circle on which all three vertices of a triangle lie is called the* **circumcircle** *of the triangle. Its center is called the* **circumcenter***.*

Definition 1.3.9. *A circle is* **inscribed in a polygon** *if each side of the polygon is tangent to the circle.*

We will be primarily interested in circles inscribed in triangles. For the triangle, we call the circle inscribed the **incircle** of the triangle and its center is called the **incenter**.

Definition 1.3.10. *Two polygons are* **similar** *if there is a matching of their vertices for which the corresponding angles are congruent and the corresponding sides are proportional.*

Theorem 1.3.9. *If two triangles have the three angles of one congruent respectively to the three angles of the other, then the triangles are similar.*

Corollary 1.3.1. *If triangles have two angles of one congruent to two angles of the other, then the triangles are similar.*

Corollary 1.3.2. *If two right triangles have an acute angle of one congruent to an acute angle of the other, then the right triangles are similar.*

Corollary 1.3.3. *Corresponding altitudes of two similar triangles have the same ratio as that of any two corresponding sides.*

Theorem 1.3.10. *If two triangles have an angle of one congruent to an angle of the other and the sides including these angles are proportional, then the triangles are similar.*

Theorem 1.3.11. *If two triangles have their corresponding sides proportional, then they are similar.*

Theorem 1.3.12. *The altitude on the hypotenuse of a right triangle forms two triangles that are similar to the given triangle and similar to each other.*

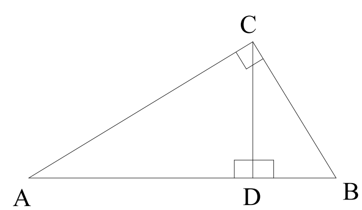


Figure 1.3.3

From Theorem 1.3.12, we have in Figure 1.3.3, that

$$\triangle ABC \sim \triangle ACD \sim \triangle CBD.$$

Theorem 1.3.12 can be used to obtain a proof for the Pythagorean theorem.

Theorem 1.3.13. ***(Pythagoras)*** *If triangle ABC is a right triangle with right angle at C, then $a^2 + b^2 = c^2$.*

Proof. In Figure 1.3.4 draw $CD \perp AB$. From Theorem 1.3.12, we have that $\triangle ABC \sim \triangle ACD \sim \triangle CBD$. Therefore, $\dfrac{a}{s} = \dfrac{c}{a}$ and $\dfrac{c}{b} = \dfrac{b}{r}$. Hence $a^2 = cs$ and $b^2 = cr$. Adding these two equations gives $a^2 + b^2 = cs + cr = c(s + r) = c^2$.

We will now examine the angles associated with arcs on a circle. Let A, B and C be points on a circle centered at O. $\angle BAC$ is called an *inscribed angle*, since it is inscribed in a circle. $\angle BOC$ is called a *central angle* and the arc BC has angular measure measured by $\angle BOC$, where the angle is measured on the same side of O as the arc. $\angle BAC$ is said to *subtend* the arc BC.

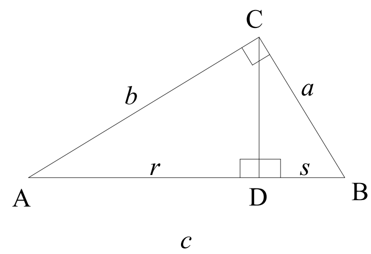


Figure 1.3.4

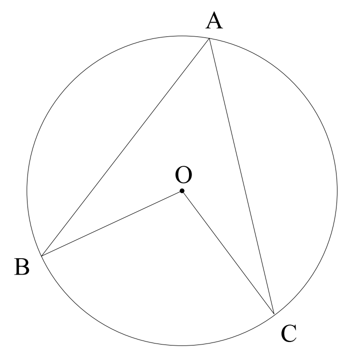


Figure 1.3.5

Theorem 1.3.14. ***(Star Trek Lemma)****. The measure of an inscribed angle is half of the angular measure of the arc it subtends.*

Proof. Let us first consider the two cases where the angle $\angle BAC$ is an acute angle. The first case we consider is where O lies inside the angle – the left-hand part of Figure 1.3.6. First we draw diameter AOD Now $OA = OB = OC = OD$, since they are all radii of the circle. $\triangle AOB$ is an isosceles triangle, since $OA = OB$. Therefore $\angle BAO = \angle OBA$. Since an exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles, we have that $\angle BOD = \angle OBA + \angle BAO = 2\angle BAO$. Similarly, using $\triangle AOC$, we have that $\angle COD = 2\angle OAC$. Combining these results gives the following and our first case is done.

$$\angle BOC = \angle BOD + \angle DOC = 2\angle BAO + 2\angle OAC = 2\angle BAC.$$

In the right-hand part of Figure 1.3.6, we have the case where the center of the circle is outside the acute angle $\angle BAC$. Our proof of this case is very similar to the previous with one significant difference. First we again note the existence of isosceles triangles because of the several radii that are drawn. In particular, $\triangle BOA$ is an isosceles triangle with $OA = OB$. Thus $\angle OAB = \angle OBA$. As in the case above, we use the fact that $\angle BOD$ is an exterior angle for the

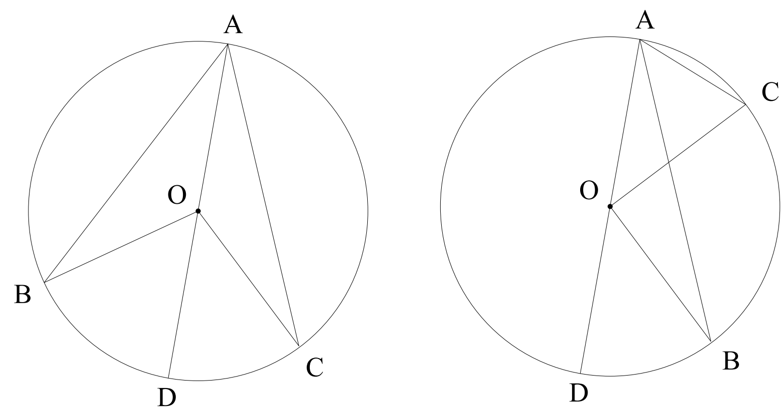


Figure 1.3.6

triangle and hence $\angle BOD = 2\angle OAB$. Furthermore, $\triangle AOC$ is isosceles with $\angle OAC = \angle OCA$. For this triangle, $\angle DOC$ is the exterior angle and we have $\angle DOC = 2\angle OAC$. Here we differ from the first case. Instead of adding, we now subtract.

$$\angle BOC = \angle DOC - \angle DOB = 2\angle OAC - 2\angle OAB = 2\angle BAC.$$

If $\angle BAC$ is obtuse, we have the following picture and the proof is identical to the first case

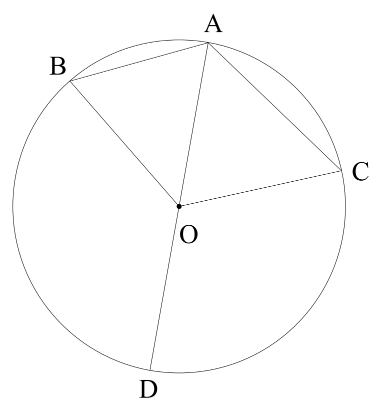


Figure 1.3.7

An immediate result of this theorem is the following: *An angle inscribed in a semicircle is a right angle.* In this case, $\angle BOC = 180°$ and this would imply that $\angle BAC$ would be a right angle. The converse of this statement is also true. *If an inscribed angle* $\angle BAC = 90°$ *then BC is a diameter of the cir-*

cle. $\angle BAC = 90°$ implies that $\angle BOC = 180°$ which says that BC is a diameter.

Another useful result is that if an angle A subtends a chord BC and arc BPC, then the angle formed by the chord and the tangent to the circle at B, which subtends arc BPC, is equal to angle A. See Figure 1.3.8.

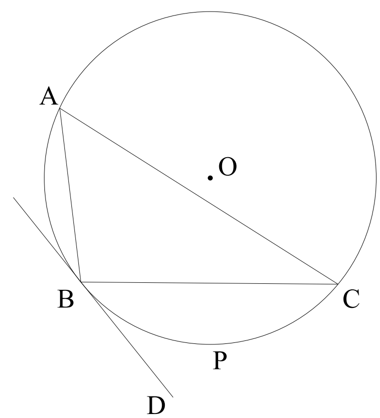


Figure 1.3.8: Angle Formed by Tangent and Chord

Definition 1.3.11. *A quadrilateral whose vertices lie on a circle is called a cyclic quadrilateral.*

Theorem 1.3.15. *If two opposite angles of a quadrilateral are both right angles, the quadrilateral is cyclic.*

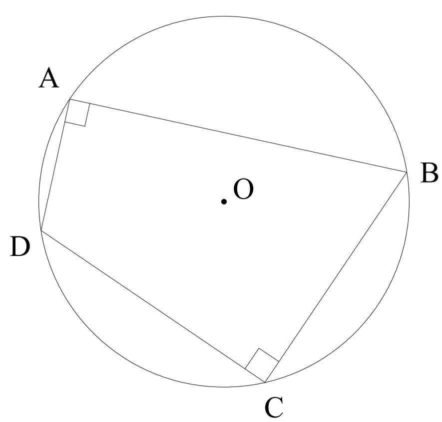


Figure 1.3.9

Proof. Exercise

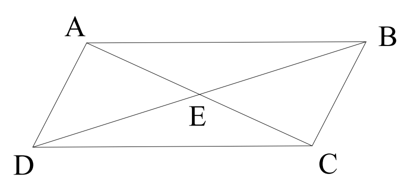
Problem Solving Strategy

Problem solving becomes more successful if one has a strategy to follow. Not everyone uses the same strategy, but most are variations of the following.

1. Read the problem carefully.
2. Draw and label a picture or diagram.
3. Identify what is given.
4. Identify what needs to be proven or constructed.
5. Mentally review all definitions, theorems or problems relating to this problem.
6. Devise a plan of attack for solving the problem.
7. If this plan works, review the solution and make note of the strategy used for future reference when solving problems of a similar type. Also see if you can find another method to solve this problem. Many problems have more than one method of solution.
8. If the plan did not work, return to step 1 and proceed through the process again and try another approach.

Example 1.3.1. *Prove that the diagonals of a parallelogram bisect each other.*

We begin by drawing a parallelogram with its diagonals and label all points.



Given: The parallelogram $ABCD$ with diagonals AC and BD.
Prove: $AE = EC$ and $BE = ED$.

What we know: Parallelograms are quadrilaterals with opposite sides equal and parallel; that is, $AB = CD$, $AD = BC$, $AB \parallel CD$ and $AD \parallel BC$. Since the diagonals are transversals for the parallel sides, the alternate interior angles are equal. For example, $\angle ADB = \angle CBD$, $\angle DAC = \angle BCA$, etc. Vertical angles are equal so that $\angle AED = \angle CEB$ and $\angle AEB = \angle CED$. We see there are similar, as well as congruent triangles available.

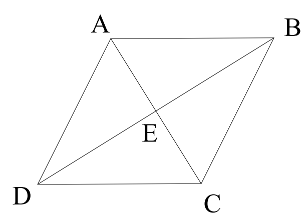
Plan of attack: We actually have two choices. We can use corresponding parts of congruent triangles are congruent. Or, we can use corresponding sides of similar triangles are proportional.

Solution: $\angle CAD = \angle ACB$, $\angle BDA = \angle DBC$ since they are alternate interior angles on the transversals AC and BD. Now $\angle EAD = \angle CAD = \angle ACB = \angle ECB$ and $\angle EDA = \angle BDA = \angle DBC = \angle EBC$ and, furthermore, $AD = CB$ since they are opposite sides of a parallelogram. Hence $\triangle AED \cong \triangle CEB$, by ASA. Thus, the corresponding sides of these triangles must be equal and $AE = EC$ and $BE = DE$, so the diagonals AC and BD bisect each other.

Note. Since $\triangle AED \sim \triangle CEB$ and corresponding parts of similar triangles are in the same ratio, we have that $\dfrac{AE}{CE} = \dfrac{DE}{BE} = \dfrac{AD}{BC}$. Since $AD = BC$, $\dfrac{AD}{BC} = 1$, so $\dfrac{AE}{CE} = \dfrac{DE}{BE} = 1$. Thus $AE = CE$ and $DE = BE$.

In the next example we show how we build on information we know to prove new results.

Example 1.3.2. *Prove that the diagonals of a rhombus are perpendicular.*



Given: The rhombus $ABCD$ with diagonals AC and BD.
Prove: $AC \perp BD$.

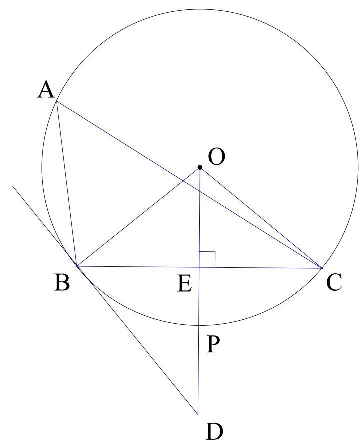
What we know: A rhombus is a parallelogram with all four sides equal. From the previous example we know that $AE = CE$ and $DE = BE$ and $\triangle AED \cong \triangle CEB$.

Plan of attack: Show all four triangles are congruent. If this is the case, then the four angles formed by the diagonals at E will be equal and since their sum is $360°$, they are all equal to $90°$ and the diagonals will be perpendicular.

Solution: From the previous example we have that the diagonals of a parallelogram, and hence the diagonals of a rhombus, bisect each other. Hence $AE = CE$ and $DE = BE$. Now the sides of a rhombus are all equal so that $AB = BC = CD = DA$. We then have $\triangle AEB \cong \triangle CEB \cong \triangle CED \cong \triangle AED$ by SSS. Thus $\angle AEB = \angle CEB = \angle CED = \angle AED$. The four angles sum to $360°$, so each is $90°$ and the diagonals will be perpendicular.

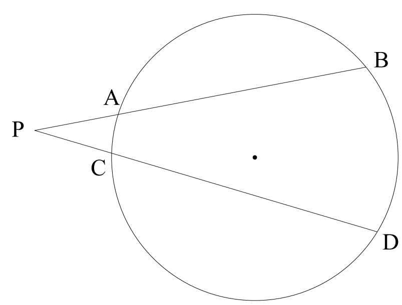
Exercises

1.3.1. *Use the following figure to prove the statement: If an angle A subtends a chord BC and arc BPC, then the angle formed by the chord and the tangent to the circle at B, which subtends arc BPC, is equal to angle A .*



That is, prove that $\angle CBD = \angle BAC$. Explain all steps.

1.3.2. *In the figure below, prove that $PA \cdot PB = PC \cdot PD$.*



1.3.3. *Given an equilateral triangle. Prove that for any point in the interior of the triangle or on a side of the triangle, the sum of the perpendicular distances from the point to the sides is constant. What is that constant?*

1.3.4. *Prove that the area of a triangle can be computed as $\frac{1}{2}$ the product of two sides and the sine of the included angle. For example, if $|ABC|$ represents the area of $\triangle ABC$, then $|ABC| = \frac{1}{2}ab\sin C$, $|ABC| = \frac{1}{2}ac\sin B$, $|ABC| = \frac{1}{2}bc\sin A$.*

1.3.5. *Use the Extended Law of Sines and one of the above area formulas for a triangle to prove that if R is the circumradius of triangle ABC, then* $|ABC| = \dfrac{abc}{4R}$. *This can also be written as* $R = \dfrac{abc}{4|ABC|}$.

1.3.6. *Use Figure 1.3.10 to give another proof for* $|ABC| = \dfrac{abc}{4R}$.

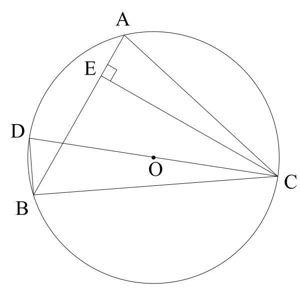


Figure 1.3.10

1.3.7. *Prove that the area of a rhombus is $\frac{1}{2}$ the product of its diagonals.*

1.3.8. *Prove that the midpoints of the sides of a quadrilateral form a parallelogram.*

1.3.9. *In Figure 1.3.11, CD is a chord of circle $O(r)$ (a circle with center O and radius r) which is perpendicular to the diameter AB. If $CE = a$ and $AE = b$, prove that* $r = \dfrac{a^2 + b^2}{2b}$.

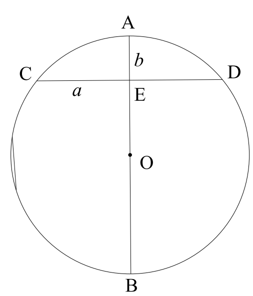


Figure 1.3.11

1.3.10. *Let $ABCD$ be a rhombus. If the radius of the circumcircle for $\triangle ABC$ is 20 units and the radius of the circumcircle for $\triangle ABD$ is 10 units, find the*

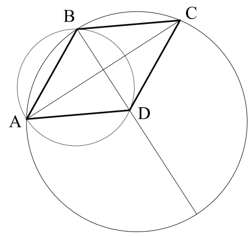


Figure 1.3.12

area of the rhombus. (Consider Figure 1.3.12.)

1.3.11. *Prove Theorem 1.3.15.*

1.3.12. *Prove the following theorem.*

Theorem. *If P is a point on the circumcircle of triangle ABC and if D, E, F are the feet of perpendiculars from P to sides BC, CA and AB, the points D, E, F are collinear. This line of collinearity is called the* Simpson line for the point P and triangle ABC.

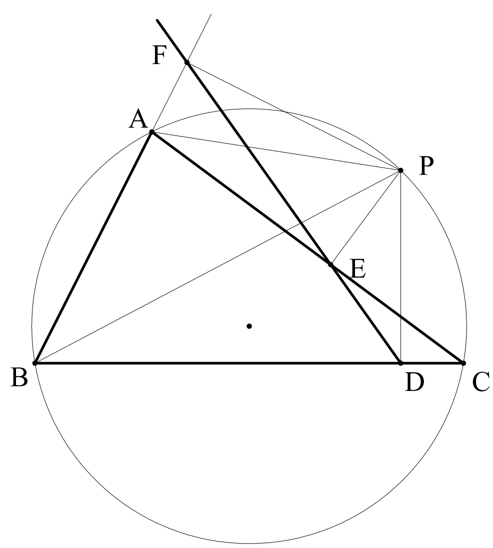


Figure 1.3.13: Simson Line

1.4 Some Basic Geometric Constructions

We need to examine how to do some basic constructions that may be required in later work. Most of the time, when we are asked to perform a construction, it is very helpful to hand draw a picture of what you are trying to construct. Of course, in order to do constructions, one must have available a straight-edge and compass. Although a drawing or construction will not, by itself, prove a result, it can often help in visualizing what one needs to do to generate a proof. Throughout the course, we will have many instances in which we need to construct a figure. We will now examine some basic constructions that we may find useful later.

Construction 1.4.1. *To divide a given segment into a given number of equal parts.*

Suppose we wish to divide a segment into five equal parts. If we recall Theorem 1.3.6, which states

Parallel lines cut off proportional segments on two transversals.

we can make use of this theorem in the following way. Suppose at one end of the given segment we draw a line of arbitrary length and on this new line we lay off five equal segments. We can then connect the end of the last segment to the end of our given line. At the other four dividing points we draw lines parallel to the end line. The result, by Theorem 1.3.6, is the given segment divided into five equal parts.

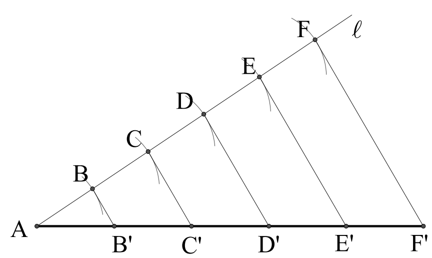


Figure 1.4.1: Dividing a Given Segment Into Five Equal Parts

In the above figure, we let A be the left endpoint of the given segment. Through A we draw a line ℓ and on ℓ we lay off five equal segments: AB, BC, CD, DE, EF. We then join point F to the endpoint F' of the given segment. Then at points $B,\ C,\ D,\ E$ we draw lines parallel to FF'. This is done by reproducing $\angle EFF'$ at each of the points $B,\ C,\ D,\ E$.

Theorem 1.4.1. *The center of a circle lies on the perpendicular bisector of any chord of the circle.*

Proof. The perpendicular bisector of a line segment is the locus of all points equidistant from the endpoints. Let AB be a chord of the given circle and let M be the midpoint of AB. The perpendicular to AB at M has the property that every point on it is equidistant from A and B. Since all points on a circle are equidistant from the center of the circle, the center O of the circle lies on the perpendicular bisector of AB.

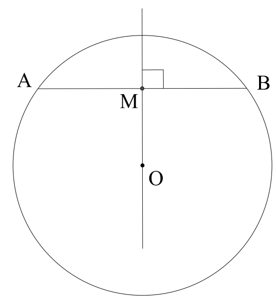


Figure 1.4.2: Perpendicular Bisector of a Chord

Construction 1.4.2. *To construct the circumcircle for a given triangle.*

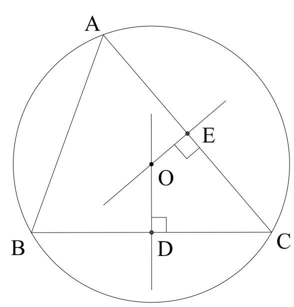


Figure 1.4.3: The Circumcircle

Since the vertices of triangle ABC lie on the circumcircle, each side of the triangle is a chord of the circle. Hence, by Theorem 1.4.1, the perpendicular bisectors of the three sides all pass through the circumcenter. All we need do then is find the point of intersection of any two of the perpendicular bisectors. This will locate the center. We find the midpoint, D, of BC and construct the perpendicular to BC at D. We then find the midpoint E of side AC and draw the perpendicular there. These two perpendiculars meet in O, the circumcenter. The circumradius is OA.

Construction 1.4.3. *To draw the tangent lines to a given circle from a given point outside the circle.*

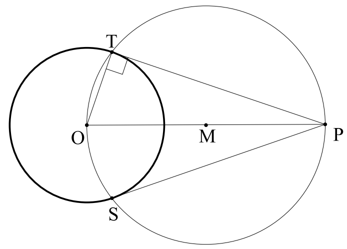


Figure 1.4.4: Tangents to a Circle

Let the circle with center O be the given circle and P the given point. Since the tangent to a circle is perpendicular to the radius drawn to the point of tangency, we need to find a point T on the given circle so that OT is perpendicular to PT. Now the locus of all points forming a right angle with lines from O and P would be a circle drawn on OP as diameter. Hence we join O and P and find the midpoint M and draw the circle with radius MO. Where this circle cuts the given circle are the points T and S. The lines PT and PS are then the required tangents.

Following are some examples of basic constructions. From the figures given, determine the method of construction.

Example 1.4.1. *To bisect a given angle.*

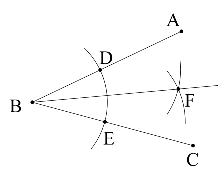


Figure 1.4.5: Angle Bisection

Example 1.4.2. *To draw a perpendicular from a given point P to a given line ℓ.*

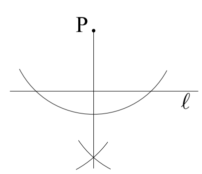


Figure 1.4.6: Perpendicular

Exercises

1.4.1. *Construct a line parallel to a given line through a given point.*

1.4.2. *Construct a circle with given center and tangent to a given line.*

1.4.3. *Construct a circle through a given point P and tangent to a given line ℓ at a given point Q.*

1.4.4. *Draw a tangent to a given circle parallel to a given line.*

1.4.5. *With a given radius construct a circle tangent to two given intersecting lines.*

1.4.6. *Construct a circle tangent to each of three lines which intersect in three distinct points.*

1.4.7. *Construct a circle having a given radius, tangent to a given circle, and passing through a given point.*

1.4.8. *Construct a circle passing through three given non-collinear points.*

1.4.9. *Six mutually tangent circles of equal radii, r, are inscribed in a larger circle of radius R, as in Figure 1.4.7 below.*

- *(a) What is the radius r of the small circles?*
- *(b) What is the area of the shaded region in Figure 1.4.8. below?*
- *(c) Construct Figure 1.4.7 and explain the construction.*

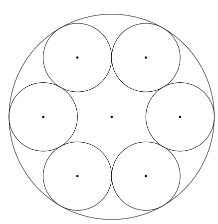


Figure 1.4.7: Figure for Part (a)

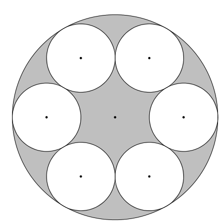


Figure 1.4.8: Figure for Part (b)

With compass and straightedge we know we can draw circles and lines; however it is possible to draw geometric figures with these tools that we might not expect possible. Consider the following "Euclidean eggs," which were constructed using a compass and straightedge.

The middle egg was constructed as follows (See Figure 1.4.10):

1. Begin with a segment AB.

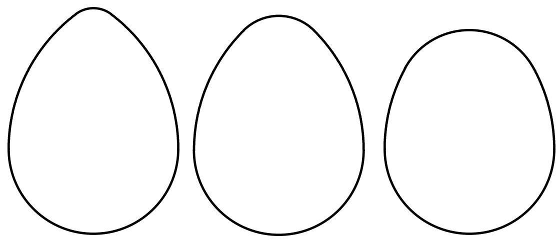


Figure 1.4.9: Euclidean Eggs

2. Construct the perpendicular bisector of AB
3. Construct circle c_1 on AB as diameter.
4. Construct circles c_2 and c_3 with radius AB on centers A and B.
5. Let O be the intersection of the perpendicular bisector of AB and circle c_1.
6. Construct lines AC and BD through O
7. Construct the circle with center O and radius OC.

The egg is the join of arcs AB, BC, CD and DA

1.4.10. *Describe how eggs one and three are constructed in Figure 1.4.9.*

1.4.11. *Prove that the Euclidean egg is a smooth curve.*

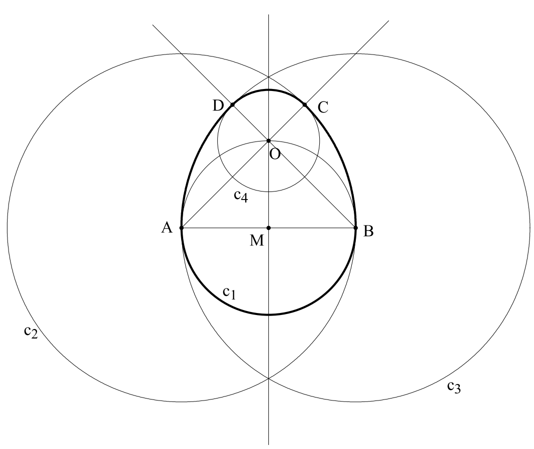


Figure 1.4.10: Construction of a Euclidean Egg

1.5 Triangle Constructions

An important part of geometry has been the ability to construct, if possible, a triangle from a given set of data. To this end, we recall, or introduce, some common notations.

- $A,\ B,\ C\ldots$ will usually denote the vertices or corresponding angles of a polygon. The upper case letters are also used to label points.
- $a,\ b,\ c\ldots$ will usually denote the sides of the polygon. In the case of a triangle, the lower case letters will denote the side opposite the vertex given by the upper case letter. Lower case letters are also used to label a line.
- $2s$ will denote the perimeter of a triangle.
- $h_a,\ h_b,\ h_c$ will denote the altitudes of a triangle drawn to the sides a, b, c respectively.
- $m_a,\ m_b,\ m_c$ will denote the medians of a triangle drawn to the sides a, b, c respectively.
- $t_a,\ t_b,\ t_c$ will denote the internal, and $t'_a,\ t'_b,\ t'_c$ the external bisectors of the angles $A,\ B,\ C$ of a triangle.

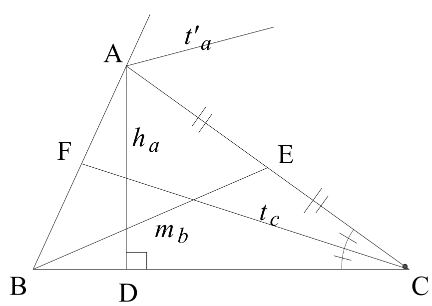


Figure 1.5.1: Labeling of a Triangle

- $R,\ r$ will denote the radii of the circumscribed and inscribed circles. For these circles, we have the terminology: *circumcircle, circumradius (R), circumcenter (O), and incircle, inradius (r), incenter (I).*

When trying to perform a construction it is worthwhile to hand draw what the end figure should be and label the information that is given. Then try to determine how you can get from the given information to the sought construction. We use the following as an example of how this can be done.

Construction 1.5.1. *Construct a triangle given side a, altitude h_a and median m_a*

We start by drawing a rough picture of a triangle with an altitude to side a and a median to side a. We then identify those parts of the triangle we know. In Figure 1.5.2, we have drawn a triangle and circled the known parts: a, h_a, m_a. We now try to determine how we might put this information to use in order to actually construct such a triangle.

Our first observation is that since we know a, we know the midpoint of a. Using that as a center and m_a as a radius we get a circle which is the locus of vertex A, since A is m_a units from the midpoint of a. We then need to determine where on that circle A lies. Since we know h_a, we know the perpendicular distance of A from a. Hence if we construct the perpendicular to a and measure off h_a we only need to draw a line parallel to a through the opposite end of the line of length h_a. Where this parallel meets the circle are the locations for A. We see that in order to have a solution we must have that $m_a \geq h_a$. There is exactly one solution if $m_a = h_a$ and two solutions if $m_a > h_a$.

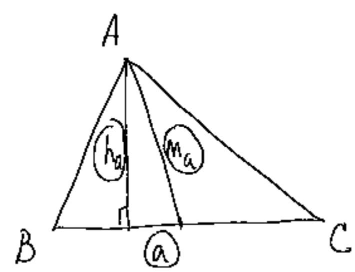


Figure 1.5.2: Sketch

Let's test our analysis. Below we are given three segments of length a, h_a and m_a. We want to construct a triangle, if one exists, that satisfy this data. We begin by drawing a line ℓ and on this line lay off

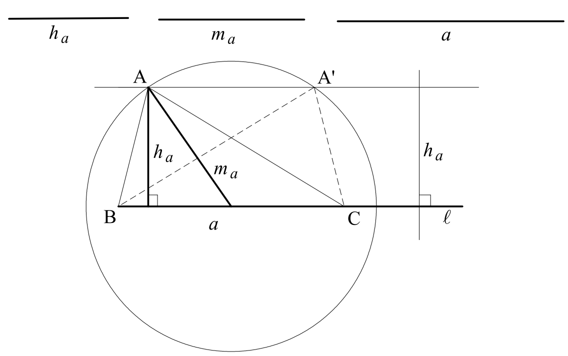


Figure 1.5.3: Triangle Construction

a segment $BC = a$. We next bisect BC. Using this midpoint as a center, we draw a circle of radius m_a. Next, at any point on ℓ we erect a perpendicular and measure h_a on it. We then draw a line parallel to BC. This line cuts the circle in two points. Each point will yield a solution. It is customary to identify multiple solutions and to label the triangle, especially the given parts, as in Figure 1.5.3.

Construction Problems Given the Perimeter or the Sum of Two Sides

Some problems seem to not have enough information to obtain a solution. One such problem is to determine how to construct a triangle when we are given the perimeter of the triangle, one angle and an additional piece of information. We begin with the following theorem.

Theorem 1.5.1. *If the perimeter and one angle of a triangle are given, the locus of the vertex of this angle is an arc of the circle in which a chord equal to the perimeter subtends an angle equal to* $90°$ *plus half the given angle.*

Proof. Let ABC, in Figure 1.5.4, be the triangle in question having the given perimeter $2s$ and the given angle A.[Recall that $s = \frac{1}{2}(a+b+c)$.] Extend the base BC to E on the left and to F on the right so that $BE = BA$ and $CF = CA$. Now triangles EBA and FCA are isosceles and $\angle EAB = E = \frac{1}{2}B$ and $\angle CAF = F = \frac{1}{2}C$. Thus we obtain

$$\angle EAF = \frac{1}{2}B + A + \frac{1}{2}C = \frac{1}{2}(A+B+C) + \frac{1}{2}A = 90° + \frac{1}{2}A.$$

The circle in question is merely the circumcircle of triangle EAF whose center is the intersection of the perpendicular bisectors of EA and FA.

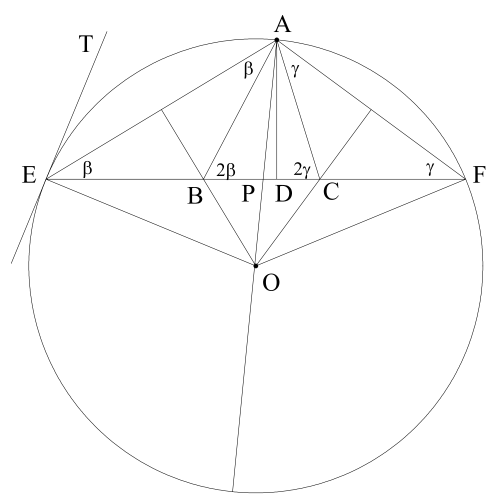


Figure 1.5.4

Since EF was constructed to be equal to $2s$ and EF subtends the known angle EAF at A, we see the locus of vertex A is indeed the arc of the circle in the above figure.

We will now look at some of the properties that the figure (Figure 1.5.4) for the above theorem gives.

We first note that $\angle EOA = 2F = C$ and $\angle FOA = 2E = B$. Since the elongation of AO is a diameter of the circle, we have that

$$\angle EAO = \frac{1}{2}(180^\circ - C) = \frac{1}{2}(A + B).$$

Similarly

$$\angle OAF = \frac{1}{2}(A + C).$$

Now

$$\begin{aligned} \angle BAO &= \angle EAO - \angle EAB \\ &= \frac{1}{2}(A + B) - \frac{1}{2}B = \frac{1}{2}A. \end{aligned}$$

This implies that AO bisects A. Hence $AP = t_a$. If R is the radius of the circle we have that $OP = R - t_a$. We further note that OB and OC lie on the perpendicular bisectors of EA and FA, respectively. In addition, the altitude of triangle EAF upon the side EF is $AD = h_a$, the altitude of triangle ABC on side BC.

Finally we observe that $\triangle OEB \cong \triangle OAB$ since $EB = AB, OE = OA$ and $OB \equiv OB$. Therefore, $\angle OEF = \angle BAO = \frac{1}{2}A$. Likewise, $\angle OFE = \frac{1}{2}A$.

Let us see how the above information can be used to create a method for constructing such triangles.

Method: Construct a triangle given the perimeter, the angle opposite the base and the altitude to the base $(2s, A, h_a)$.

Solution. Let triangle ABC, in the figure below, be the required triangle. Extend BC on both sides and lay off $BE = BA$ and $CF = CA$. Thus $EF = 2s$, the perimeter of triangle ABC.

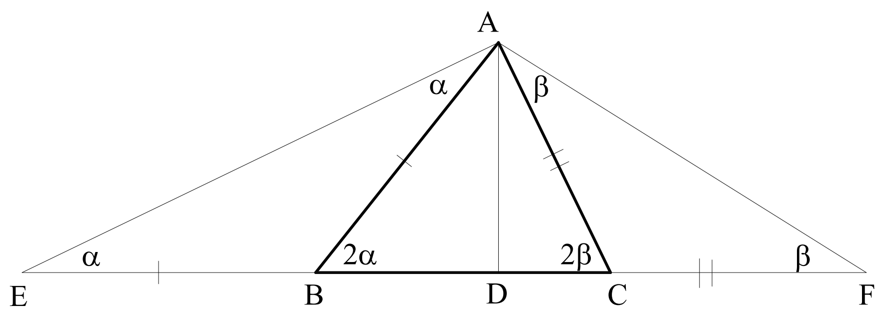


Figure 1.5.5

The triangles EAB and FCA are then isosceles triangles and we have

$$\angle E = \angle EAB = \frac{1}{2}\angle ABC = \frac{1}{2}\angle B, \quad \angle F = \angle FAC = \frac{1}{2}\angle ACB = \frac{1}{2}\angle C.$$

Hence

$$\angle EAF = \frac{1}{2}\angle B + \angle A + \frac{1}{2}\angle C = \frac{1}{2}(\angle A + \angle B + \angle C) + \frac{1}{2}\angle A = 90^\circ + \frac{1}{2}\angle A$$

and therefore $\angle EAF$ is a known angle. The altitude of triangle ABC is also the altitude of triangle AEF. Thus in triangle AEF we know the base $EF = 2s$, the opposite angle $\angle EAF = 90^\circ + \frac{1}{2}\angle A$, and the altitude $AD = h_a$. Hence this triangle may be constructed. The vertex A belongs to the required triangle ABC. Since $BA = BE$ and $CA = CF$, the vertices B and C are the intersections of the perpendicular bisectors of EA and FA, respectively, with EF.

Note: This problem may have two, one or no solutions. In the two solution case, the triangles are symmetric with respect to the perpendicular bisector of EF.

Example 1.5.1. *Construct a triangle given the perimeter, the angle opposite the base and the altitude to the base $(2s, A, h_a)$.*

Solution.

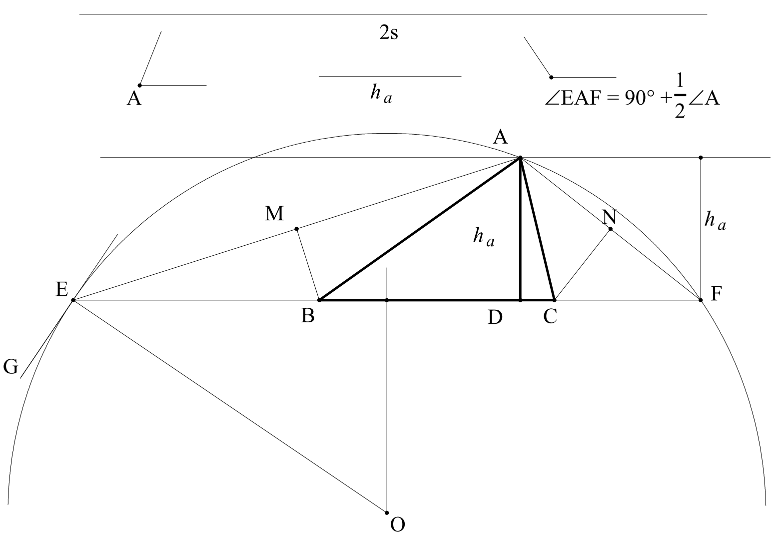


Figure 1.5.6

In Figure 1.5.6, we start by constructing $EF = 2s$. Then at E we construct an angle ($\angle FEG$) equal to $90° + \frac{1}{2}\angle A$. At E we construct the perpendicular to EG and also the perpendicular bisector of EF, where these meet is the center of a circle that is the locus of $\angle EAF$, since it is the circumcircle of triangle EAF. At F construct a perpendicular of height h_a and at its upper extremity construct a parallel to EF. Where this line cuts the circumcircle is the location of A (there will be two in this case and they will be symmetric about the perpendicular bisector of EF). Join E with A and F with A. The perpendicular bisector of EA meets EF in the vertex B and the perpendicular bisector of AF meets EF is the vertex C, giving the required triangle ABC.

We next look at the construction of a triangle given the sum of two of the sides. We first look at the method and then an example.

Method: Construct a triangle given the base, the angle opposite, and the sum of the other two sides $(a, A, b + c)$.

Solution. Let triangle ABC, in the figure below, be the required triangle. Extend BA and lay off $AD = AC$.

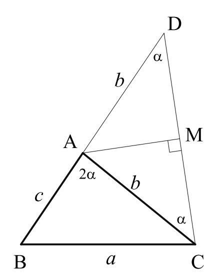


Figure 1.5.7

Then triangle ACD is an isosceles triangles and we have $\angle D = \angle ACD = \frac{1}{2}\angle BAC$. Thus, in triangle BDC we know the base $BC = a$, the side $BD = b+c$, and $\angle D = \frac{1}{2}\angle A$. Hence this triangle may be constructed. The vertices B and C belong to the required triangle and the third vertex, A, is the intersection of the perpendicular bisector of DC with BD.

Note: This problem is not possible unless $a < (b + c)$.

Example 1.5.2. *Construct a triangle given the base, the angle opposite, and the sum of the other two sides $(a, A, b + c)$.*

Solution.
In the figure above, we start by constructing $BC = a$. Then at B we construct an angle ($\angle CBE$) equal to $\frac{1}{2}\angle A$. At B we construct the perpendicular to BE and also the perpendicular bisector of BC, where these meet is the center of a circle that is the locus of $\angle D$,since it is the circumcircle of triangle DBC. With

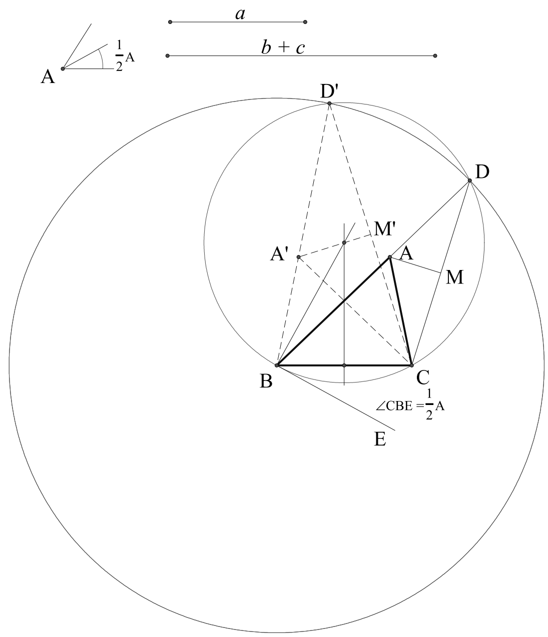


Figure 1.5.8

B as center, draw circle with radius $b+c$. Where this intersects the circumcircle is the point D. The perpendicular bisector of DC determines the vertex A of the sought triangle ABC. **Note**: The two solutions are symmetric about the perpendicular bisector of BC.

Exercises

1.5.1. *Construct a triangle when given:* a, b, c.

1.5.2. *Construct a triangle when given:* A, b, c.

1.5.3. *Construct a triangle when given:* A, B, c.

1.5.4. *Construct a triangle when given:* A, c, h_c.

1.5.5. *Construct a triangle when given:* b, c, m_c.

1.5.6. *Construct a triangle when given:* A, b, t_a.

1.5.7. *Construct a triangle when given:* h_a, t_a, A.

1.5.8. *Construct a triangle when given:* h_a, m_a, a.

1.5.9. *Construct a triangle when given: B, h_a, m_a.*

1.5.10. *Construct a triangle when given: A, a, m_a.*

1.5.11. *Construct a triangle when given: A, a, h_a.*

1.5.12. *Construct a triangle when given: A, a, h_c.*

1.5.13. *Construct a triangle given the angle at B, the altitude h_c, and the sum of two sides $b + c$.*

1.5.14. *Construct a triangle given the base a, the altitude h_c, and the sum of the other two sides $b + c$.*

1.5.15. *Construct a triangle given the angles at A and B, and the sum of two sides $b + c$.*

1.5.16. *Construct a triangle given the perimeter, $2s$, and the angles A and B.*

1.5.17. *Construct a triangle given the perimeter, $2s$, the altitude h_a and angle B.*

Historical Note.

The compass we use today is much different than the compass used in Euclid's time. The Euclidean compass, also known as the collapsing compass, did not allow for the transferring of distances. The Euclidean compass would collapse when lifted from the drawing surface. To move a given length from one position to another required some significant effort. For example, see Proposition 2 in Section 1.1. With the modern compass, we would just measure off the line with the compass and, with that fixed radius, take the line segment anywhere we wished.

The constructions we do with relative ease (once we determine how it is to be done) required much more effort with the collapsing compass. Think about the examples and exercise given above. How would you bisect an angle, or draw a perpendicular, if your compass lost its radius when you lifted it off the paper?

Pretend you have a collapsing compass; that is, you cannot transfer distances with the compass. Try these exercises.

1.5.18. *Bisect a given angle.*

1.5.19. *Construct a perpendicular from a given point to a given line.*

1.5.20. *Construct a parallel through a given point to a given line.*

1.5.21. *Given points A, P, B. At B construct a circle with radius AP.*

1.6 Constructible Numbers

For the Greek geometers the only numbers that existed were numbers that could be constructed using only a compass and straightedge. With a compass and straightedge we can construct the sum, difference, product and quotient of two numbers as well as the squareroot of a number. We do these with the modern compass. All numbers depend on a given unit length.

Constructing the number $a + b$.

To construct the sum $a+b$ on a line m lay off segments $AB = a$ and $BC = b$. The resulting segment $AC = a + b$.

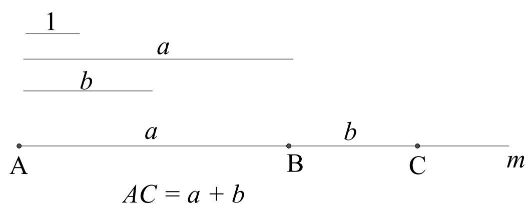


Figure 1.6.1: $a + b$

Constructing the number $a - b$.

To construct the difference $a - b$ on a line m lay off segments $AB = a$ and $BC = b$, from B toward A. The resulting segment $AC = a - b$.

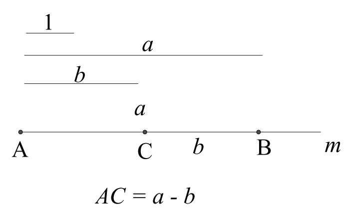


Figure 1.6.2: $a - b$

Constructing the number $a \cdot b$.

Let two lines m and m' intersect at a point A. On line m lay off segment $AB = 1$ and segment $AC = a$. On line m' lay off segment $AD = b$. Connect points B and D. At point C duplicate $\angle CBD$ and where the terminal side of this angle meets m' call E. Line $AE = a \cdot b$. See Figure 1.6.3

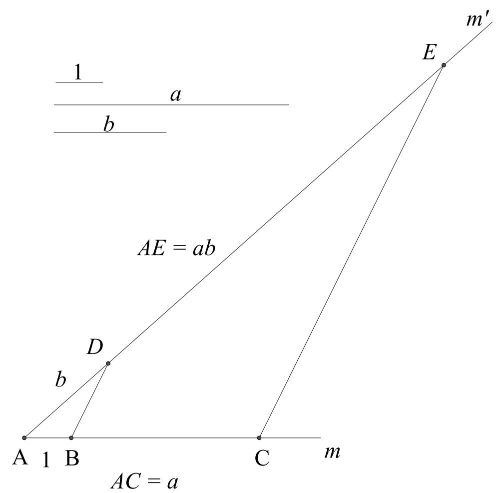


Figure 1.6.3: $a \cdot b$

Proof. $\triangle ABD \sim \triangle ACE$, since they are equiangular. Hence, $\dfrac{AD}{AB} = \dfrac{AE}{AC}$. That is, $\dfrac{AE}{a} = \dfrac{b}{1}$ and $AE = a \cdot b$.

Constructing the number $\dfrac{a}{b}$.

Let two lines m and m' intersect at a point A. On line m lay off segment $AB = 1$ and segment $AC = b$. On line m' lay off segment $AD = a$. Connect points C and D. At point B duplicate $\angle C$ and where the terminal side of this angle meets m' call E. Line $AE = \dfrac{a}{b}$. See Figure 1.6.4

Proof. $\triangle ABE \sim \triangle ACD$, since they are equiangular. Hence, $\dfrac{AD}{AC} = \dfrac{AE}{AB}$. That is, $\dfrac{AE}{1} = \dfrac{a}{b}$ and $AE = \dfrac{a}{b}$.

Although the early Pythagorean Greeks were restricting their concept of number to the rational numbers, it became evident to them that there did , by necessity, exist other numbers. They soon discovered the method of constructing the square root of a rational number. However, the construction of other roots could not in general be done. Roots which were powers of two could be constructed by repeated constructions of square roots.

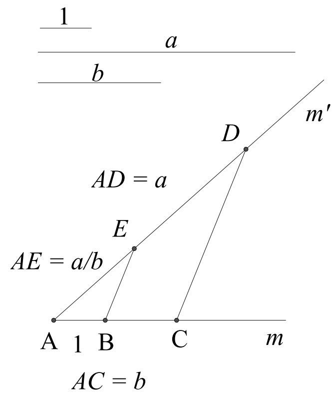


Figure 1.6.4: $\frac{a}{b}$

Constructing the number $\sqrt{a}$.

On a line m lay off segments $AB = a$ and $BC = 1$. Bisect AC getting point O. Using O as a center, construct the semicircle with diameter AC. At B erect a perpendicular meeting the semicircle at point D. The segment $BD = \sqrt{a}$. See Figure 1.6.5

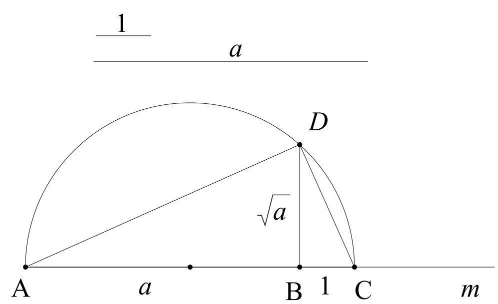


Figure 1.6.5: $\sqrt{a}$

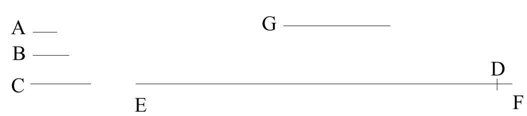
Proof. Draw lines AD and DC. Since $\angle ADC$ is inscribed in a semicircle, it is a right angle and rt$\triangle ABD \sim$ rt$\triangle DBC$. Therefore $\dfrac{BD}{AB} = \dfrac{BC}{BD}$ and $BD^2 = AB \cdot BC = a \cdot 1 = a$. Thus, $BD = \sqrt{a}$.

Now any number that can be represented as a combination of additions, subtractions, multiplications, divisions and squareroots can be constructed. Since $\sqrt[4]{3} = \sqrt{\sqrt{3}}$, the fourth root of a number is constructible. However, the cube root of a number is not constructible. It is for this reason that the general angle cannot be trisected using straight-edge and compass. In Euclid's discussion of arithmetic problems (number theory) in *The Elements*, all numbers are represented by line segments.

The following proposition, from *The Elements*, demonstrates how the Greek's dealt with problems in number theory through geometry.

Proposition 20. *Prime numbers are more than any assigned multitude of prime numbers.*

Note. This is Euclid's way of saying that the number of primes is infinite.



Let $A,\ B,\ C$ be the assigned numbers; I say that there are more prime numbers than $A,\ B,\ C$.

For let the least number measured by $A,\ B,\ C$ be taken, and let it be DE;

This means that DE is the product of $A, B,$ and C.

let the unit DF be added to DE.

Then EF is either a prime or not.

First let it be prime; then the numbers A, B, C, EF have been found which are more than $A,\ B,\ C$.

Next, let EF not be prime; therefore it is measured by some prime number. (VII, 31)

Prop. 31 of Book VII: Any composite number is measured by some prime number.

Let it be measured by the prime number G. I say that G is not the same with any of the numbers $A,\ B,\ C$. For, if possible, let it be so. Now $A,\ B,\ C$ measure DE. Therefore G will also measure DE. But it also measured EF. Therefore G, being a number, will measure the remainder, the unit DF; which is absurd.

Therefore G is not the same with any of the numbers $A,\ B,\ C$ and by hypothesis, it is prime.

Therefore the prime numbers A, B, C, G have been found which are more than the assigned multitude A, B, C. Q.E.D.

Exercises

In the following exercises, it may be assumed that given a unit length, other integer lengths are obtainable. Therefore, lengths of 2, 3, 5, 7 and 10 can be assumed.

1.6.1. *Construct the numbers* $\frac{5}{2}$ *and* $\sqrt{7}$.

1.6.2. *Construct the number* $\sqrt{\frac{5\sqrt{3}}{2}}$.

1.6.3. *Construct the number* $\sqrt[4]{10}$.

1.6.4. *There is an interesting way to construct the square root of a positive integer. In the following figure, if* $A_1B_1 = 1$*, show that*

(a) $A_2B_2 = \sqrt{2}$.

(b) $A_3B_3 = \sqrt{3}$.

(c) $A_5B_5 = \sqrt{5}$.

(d) Describe, in detail, this method of constructing $\sqrt{n}$.

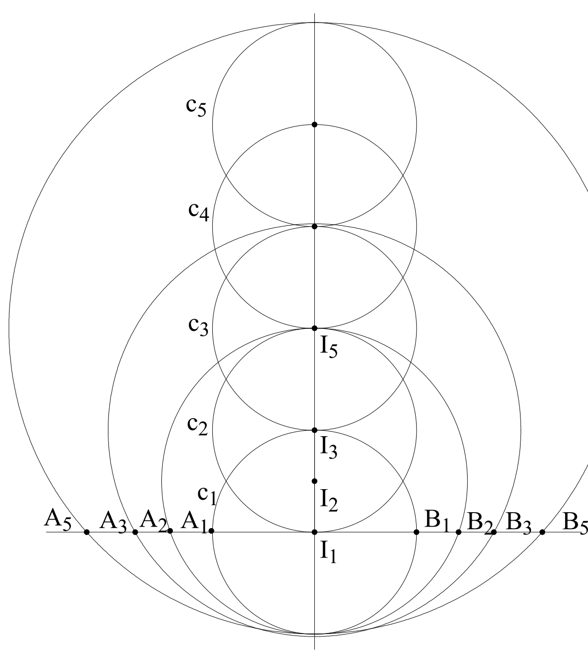


Figure 1.6.6

1.7 The Pythagorean Theorem

There are many proofs for the Pythagorean Theorem, which states:

> *Suppose a right angle triangle ABC has a right angle at C, hypotenuse c and sides a, and b. Then*
>
> $$c^2 = a^2 + b^2.$$

We will present here a few of the proofs that have been presented at various times in history. The first is believed, by some, to be close to what the Pythagoreans may have produced as a proof. Since the Pythagoreans left no written records, this is an assumption only. The proof we give uses notation that was not available to the Pythagoreans.

Proof 1 (Pythagoras). Let the right triangle have sides a and b and hypotenuse c. Consider the following figure

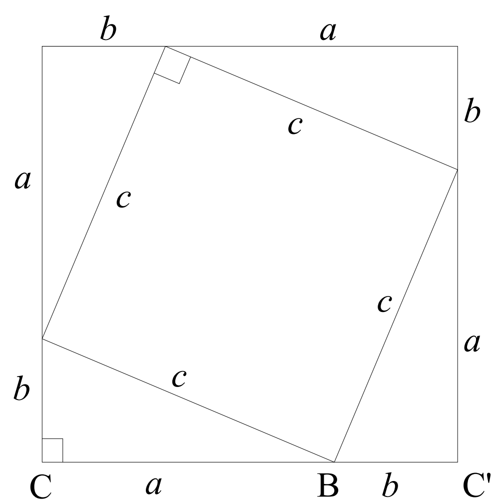


Figure 1.7.1

This is a square with side $a+b$ that has been partitioned by inscribing a square of side c in it. Since the angles at A and B sum to $90°$ and the angle $CBC' = 180°$, the angle at the vertex of the inscribed quadrilateral is $90°$ and the quadrilateral is indeed a square. Now the area of the large square can be computed in two distinct ways–first as a square of side $a+b$, and second as the area of the smaller square plus the four triangles. The first computation yields

$$(a + b)^2 = a^2 + 2ab + b^2.$$

In the second case each of the triangles has area $\frac{1}{2}ab$ and we have

$$c^2 + 4 \cdot \left(\frac{1}{2}ab\right) = 2ab + c^2.$$

Since we are computing the same area in each case, we have that

$$a^2 + 2ab + b^2 = 2ab + c^2,$$

which reduces to the desired result that $c^2 = a^2 + b^2$.

NOTE. The Pythagoreans would have given a purely geometric argument rather than an algebraic one.

This next proof is similar to the one given by Euclid in his *Elements*.

Proof 2 (Euclid). In the figure below, let ABC be our given triangle. On side AC construct square $ACDE$, on side BC construct square $BCHK$ and on the hypotenuse AB construct the square $ABFG$. We must show that the area of square $ABFG$ is equal to the area of square $ACDE$ plus the area of square $BCHK$.

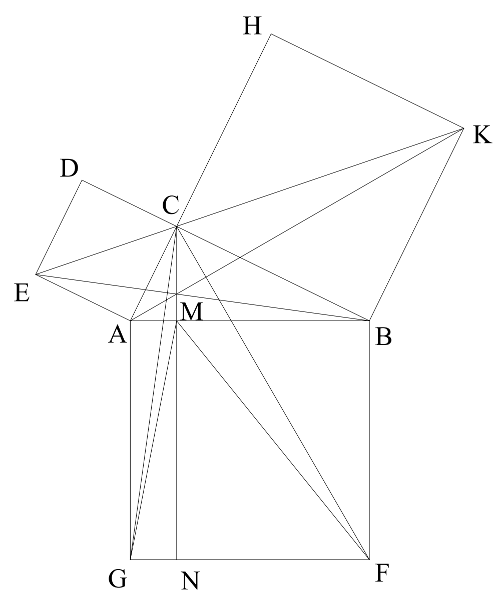


Figure 1.7.2

Draw diagonal EC of square $ACDE$. $\triangle|AEC| = \triangle|AEB|$ since they have the same base AE and their altitudes are equal to AC. (Note. We use $\triangle|AEC|$ to represent the area of triangle AEC.) We next note that, using the SAS theorem for congruent triangles, $\triangle AEB \cong \triangle ACG$, since $AE = AC, AB = AG$ and $\angle BAE = \angle GAC$ (both are equal to $90° + \angle BAC$). Therefore $\triangle|AEB| = \triangle|ACG|$. Construct $CM \perp AB$ and extend it to meet FG at N. Then $\triangle|ACG| = \triangle|AMG|$, since they share the same base, AG, and both have an altitude equal to AM. Thus, $\triangle|AEC| = \triangle|AMG|$. But $\triangle|AEC| = \frac{1}{2}$area of square$ACDE$ and $\triangle|AMG| = \frac{1}{2}$area of rectangle$AMNG$. Hence area$ACDE$ = area$AMNG$.

By a similar argument, we can show that $\triangle|BCK| = \triangle|BMF|$ and therefore area$BCHK$ = area$BMNF$. Since the area of square $ABGF$ = area$AMNG$ + area$BMNF$, we have that, the square on the hypotenuse is equal to the sum of the squares on the other two sides; that is, $c^2 = a^2 + b^2$ where $c = AB, b = AC$ and $a = BC$.

Thâbit ibn Qurra (a.k.a. Thâbit ibn Korra) (ca. 836 - 901) gave a proof for the Pythagorean Theorem by calculating the area of the figure below in two ways.

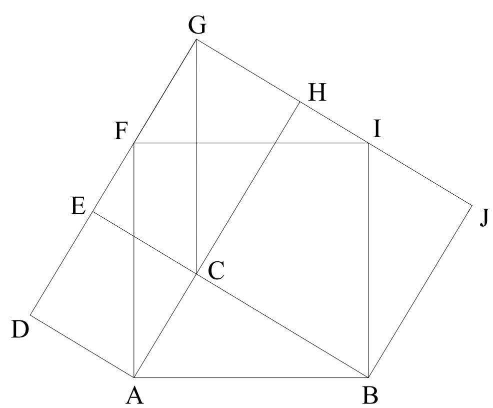


Figure 1.7.3

Proof 3 (Qurra). In the figure above, ABC is the given triangle with sides a, b and hypotenuse c. Square $ACED$ has sides of length b, square $BCHJ$ has sides of length a and square $ABIF$ has sides of length c. Now triangles CEG, CHG and IJB are all congruent to triangle ACB In the figure above, $JH = a$ and $HG = b$ so the rectangle $BJGE$ has area $a(a + b)$. Using this rectangle and the square $ACED$ with area b^2 and the triangle ACB with area $\frac{1}{2}ab$ we see that the area of the figure is

$$a(a + b) + b^2 + \frac{1}{2}ab = a^2 + b^2 + \frac{3}{2}ab.$$

On the other hand, the figure consists of the square $ABIF$ with area c^2 and three triangles (BJI, GCE and AFD) each with area $\frac{1}{2}ab$. The area of the figure using this square and triangles is

$$c^2 + 3\left(\frac{1}{2}ab\right).$$

Thus, equating the two areas gives

$$c^2 + 3\left(\frac{1}{2}ab\right) = a^2 + b^2 + \frac{3}{2}ab.$$

Hence, $c^2 = a^2 + b^2$.

The Indian mathematician Bhaskara II (ca. 1114 - 1185) gave a dissection proof that he considered self-explanatory. I have added some labels to make sure it is clear.

Proof 4 (Bhaskara).

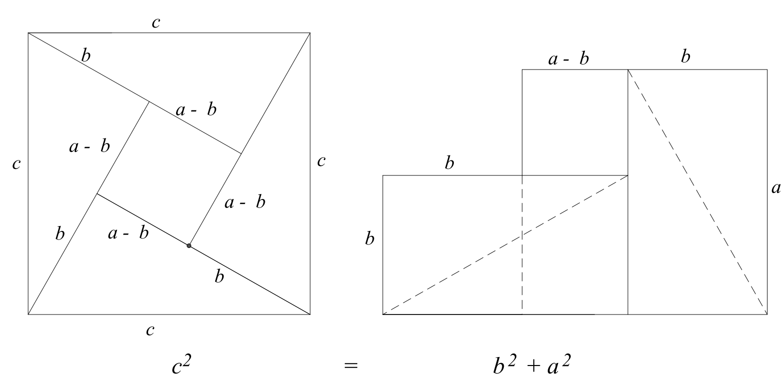
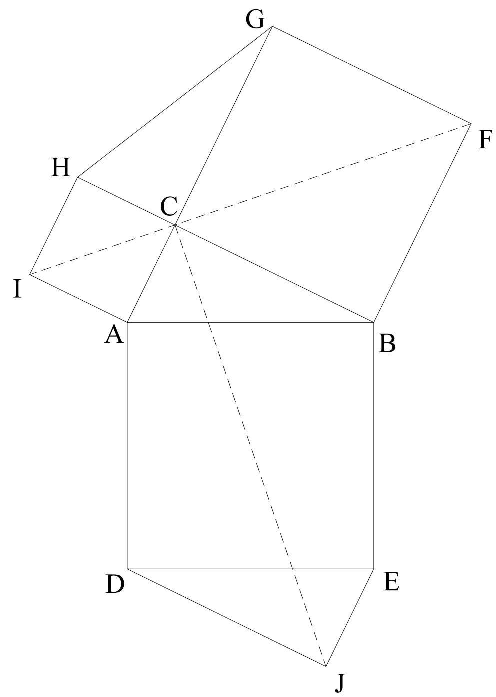


Figure 1.7.4

The figure on the left can be disassembled and reassembled as the figure on the right. The figure on the left is a square of side c and has area c^2, whereas the figure on the right can be seen to be two squares of areas a^2 and b^2. Since the figure on the right was constructed from the parts of the figure on the left, we have that $c^2 = a^2 + b^2$.

Leonardo Da Vinci (1452 - 1519) also gave a diagram proof that he felt was self-explanatory. His proof consisted of the following drawing. We will supply some explanation.

Proof 5 (da Vinci). In the figure above, the line IF divides the hexagon $ABFGHI$ into two equal quadrilaterals. If quadrilateral $ABFI$ is rotated about

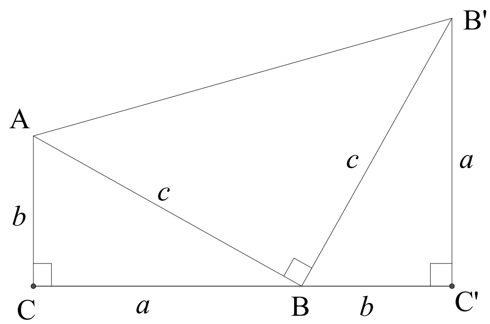


A, it will coincide with quadrilateral $CADJ$. If this quadrilateral is then rotated about B, it will coincide with $CBEJ$. Therefore the hexagons $ABFGHI$ and $CADJEB$ have the same area. The area of hexagon $ABFGHI = a^2 + b^2 + 2\left(\frac{1}{2}ab\right)$ and the area of hexagon $CADJEB = c^2 + 2\left(\frac{1}{2}ab\right)$. Thus, $c^2 = a^2 + b^2$.

The next proof we present is due to James Garfield, who later became President Garfield. In 1863 he was elected to the United States House of Representatives. He succeeded in gaining re-election every two years up through 1878. During his tenure as a member of the House of Representatives he came up with a proof of the Pythagorean theorem. Later, in 1880, he was elected President of the United States. He took office in March of 1881 and was assassinated in September of 1881. He served for only six months. This was the second shortest term in U.S. history. Only William Henry Harrison served for a shorter period. It is an interesting note that Garfield was ambidextrous and he could simultaneously write in Greek with one hand and in Latin with the other.

Proof 6 (Garfield). In the figure below, we draw triangle ABC and extend BC by b units in order to construct triangle $BC'B'$ which is congruent to triangle ACB. Connect A to B'. We now have a trapezoid $ACC'B'$.

Since $\angle CBA + \angle C'BB' = 90°$, we have that $\angle ABB' = 90°$ and triangle ABB'



is a right triangle. Now the area of the trapezoid is given by

$$\frac{1}{2}(a+b)(a+b).$$

On the other hand, the area of the three triangles making up the trapezoid is

$$\frac{1}{2}c^2 + 2\frac{1}{2}ab.$$

Since both represent the same area, we have

$$\begin{aligned}
\frac{1}{2}(a+b)(a+b) &= \frac{1}{2}c^2 + 2\frac{1}{2}ab \\
\frac{1}{2}(a^2+2ab+b^2) &= \frac{1}{2}c^2 + ab \\
a^2+2ab+b^2 &= c^2+2ab \\
a^2+b^2 &= c^2
\end{aligned}$$

A pretty slick proof.

Our final proof is one for which I do not know the origin. It is colorful and again disassembles a square of side c and reassembles it into two squares of sides a and b.

Proof 7 (Colorful). The figure below is self-explanatory, but we will give a justification. Since this is a black and white printing, each color is labeled: r =red, $b =$ blue, $g =$ green, $v =$ violet and $y =$ yellow.

We need to verify that $\triangle EFG \cong \triangle KJI, \triangle BDE \cong \triangle ALK$ and $\triangle ACB \cong \triangle IJG$ Since $\triangle IJG$ is a copy of $\triangle ACB$ the later congruence holds. For the first pair of triangles we need to show that $EF = JK$ and $FG = JI$. We are given that $FG = JI = b$. Since $FG \parallel HB$, we have that $\angle DBE = \angle FGE$ and since angles DEB and FEG are vertical angles, they are equal. Therefore $\triangle GFE \sim \triangle BDE$ and $\dfrac{DE}{FE} = \dfrac{DB}{FG}$. But $FG = HD = b$ so $DB = a - b$ and

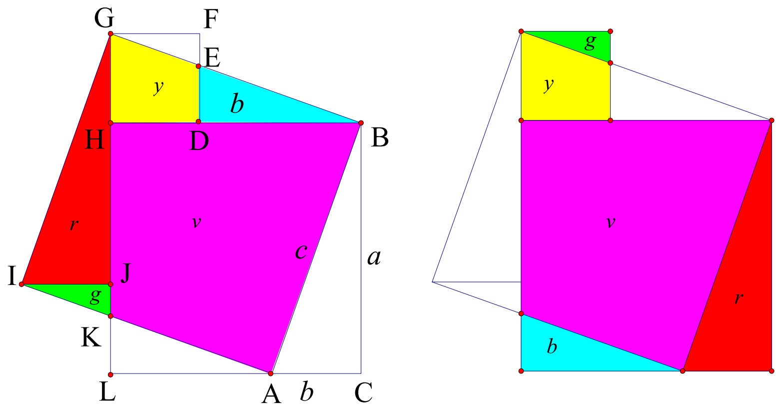


Figure 1.7.5

$DE = b - FE$. Hence $(b - FE)b = FE(a - b)$ so $FE = \dfrac{b^2}{a}$ and $DE = b - \dfrac{b^2}{a}$. By a similar argument $JK = \dfrac{b^2}{a}$ and $KL = b - \dfrac{b^2}{a}$. Thus, by SSS, $\triangle EFG \cong \triangle KJI$. Since we have all the required sides, it easily follows that $\triangle BDE \cong \triangle ALK$. The mapping in the above figure is then valid and we see that $c^2 = a^2 + b^2$.

1.8 The Incircle and Excircles

Theorem 1.8.1. *The angle bisectors of a triangle intersect in a common point I, called the* incenter*, which is the center of the unique circle inscribed in the triangle. This circle is called the* incircle *of the triangle.*

***Proof*.** In Figure 1.8.1 below, let AD, BE, CF be the angle bisectors. Let AD and BE intersect in a point I.

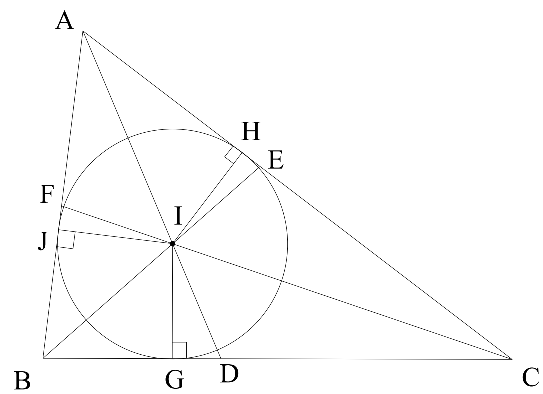


Figure 1.8.1: The Incircle

Let IJ be the perpendicular from I to side AB, let IH be the perpendicular to side AC and let IG be the perpendicular to side BC of triangle ABC. Now $AI = AI$ and since AD is the angle bisector of $\angle BAC$, $\angle JAI = \angle HAI$. Furthermore, angles IJA and IHA are right angles, by construction. Therefore, $\triangle AIJ \cong \triangle AIH$ and, hence, $IJ = IH$. Similarly, using triangles BIJ and BIG, we can show that $IJ = IG$ and hence $IH = IG$ and I lies on the angle bisector of $\angle ACB$. Thus, the three angle bisectors are concurrent.

We must now show that I is the center of a circle inscribed in triangle ABC. The three points G, H, J determine a unique circle, since they are noncollinear and three noncollinear points determine a unique circle. The fact that this circle is inscribed in triangle ABC follows from the fact that IG, IH, IJ are radii of the circle and are perpendicular to the sides BC, AC, AB at the points G, H, J and hence are tangents to the circle. □

Using arguments very much like the above, we can establish the existence of three circles which lie outside of the triangle and that are tangent to the sides of the given triangle. In this case, the circles will be tangent to two extended sides and one regular side. We call these circles *excircles* and we designate their centers by I_a, I_b, I_c. The subscript designation comes from the non-extended side the circle is tangent to. The centers are determined by the intersection of one internal angle bisector and the two external angle bisectors of the other two angles. That is, I_a is the intersection of the internal angle bisector of the angle at A and the external angle bisectors at B and C. The figure below shows the incircle and the three excircles for triangle ABC. By convention, we use r

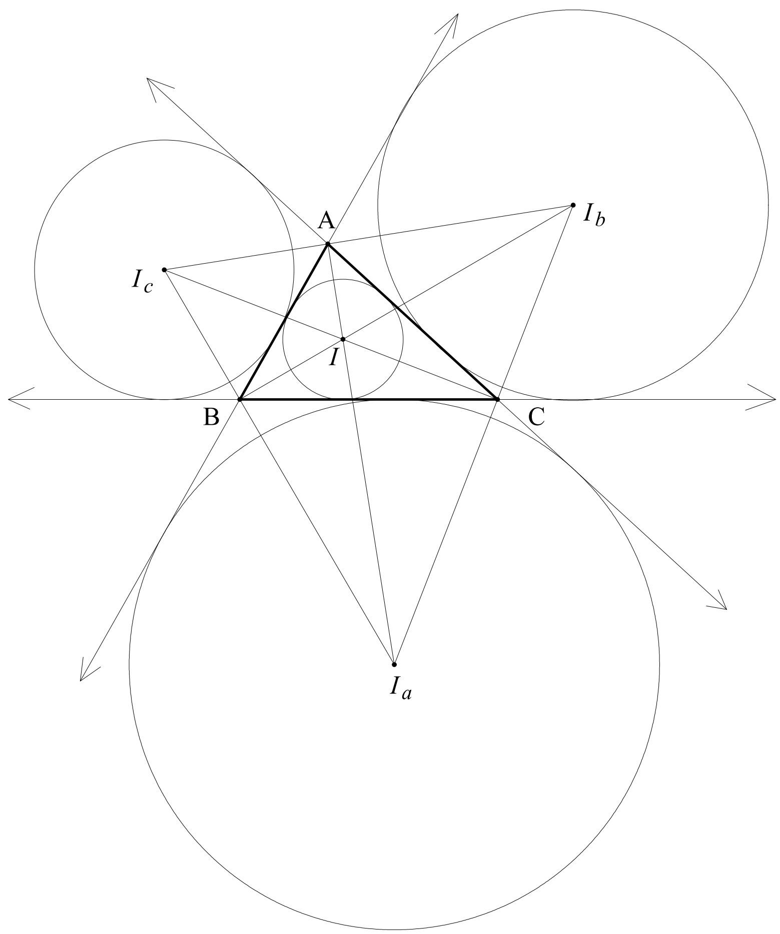


Figure 1.8.2: The Three Excircles for Triangle ABC

to represent the radius of the incircle and r_a, r_b, r_c to represent the radii of the three excircles.

Theorem 1.8.2. *Let r be the radius of the incircle of triangle ABC and let $s = \frac{1}{2}(a+b+c)$ be the semiperimeter of triangle ABC. Then*

$$|\triangle ABC| = rs.$$

[We use the notation $|\triangle ABC|$ to represent the area of triangle ABC.]

Proof. Let $BC = a, AC = b$ and $AB = c$. Let D, E, F be the feet of the perpendiculars to the sides from the incircle center I. We can subdivide

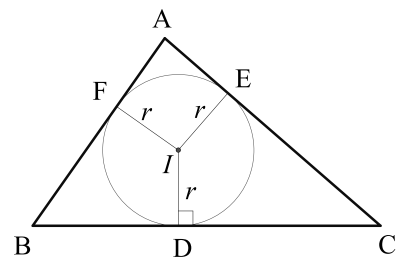


Figure 1.8.3

$\triangle ABC$ into three triangles: $\triangle BIC, \triangle CIA$ and $\triangle AIB$. The areas of these are $|\triangle BIC| = \frac{1}{2}ar$, $|\triangle CIA| = \frac{1}{2}br$, and $|\triangle AIB| = \frac{1}{2}cr$. Thus,

$$|\triangle ABC| = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = r[\frac{1}{2}(a+b+c)] = rs.$$

□

Theorem 1.8.3. (The Law of Cosines) *For any triangle ABC, we have*

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Proof. Consider a triangle ABC with altitude AD, as in Figure 1.8.4. Applying the Pythagorean theorem to triangle ABD, we have

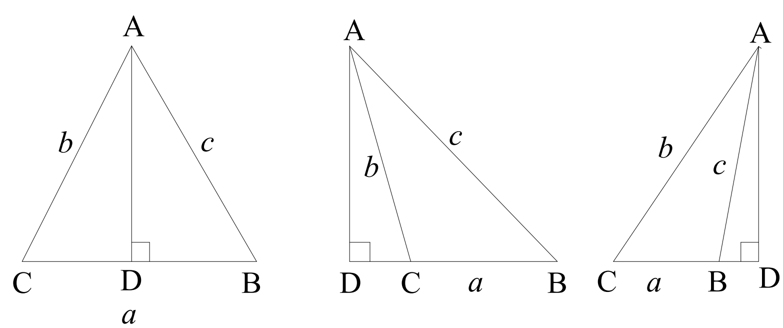


Figure 1.8.4

$$c^2 = AD^2 + DB^2.$$

But $AD = b\sin C$ in all triangles and $DC = b\cos C$ in the triangle on the left, but $DC = -b\cos C$ for the triangle in the middle. In both cases, $DB = a - DC$, or $DB = a - b\cos C$. However, in the triangle on the right $a - DC < 0$, so $DB = |a - b\cos C|$. Thus in all cases

$$\begin{aligned} c^2 &= (b\sin C)^2 + (a - b\cos C)^2 \\ &= b^2\sin^2 C + a^2 - 2ab\cos C + b^2\cos^2 C \\ &= a^2 + b^2(\sin^2 C + \cos^2 C) - 2ab\cos C \\ &= a^2 + b^2 - 2ab\cos C \end{aligned}$$

□

Note that the Law of Cosines yields the Pythagorean theorem when $\angle C = 90°$. We further note that if $\angle C < 90°$, then $\cos C > 0$ and $c^2 < a^2 + b^2$; but if $\angle C > 90°$, then $\cos C < 0$ and $c^2 > a^2 + b^2$.

From the proof of the Law of Cosines, we note that since $AD = b\sin C$ and AD is the altitude of triangle ABC, $|\triangle ABC| = \frac{1}{2}ab\sin C$. In fact, for any triangle, if we know two sides and the included angle, then the area of the triangle is given by *one-half the product of the two sides and the sine of the included angle.* That is,

$$\begin{aligned} |\triangle ABC| &= \frac{1}{2}ab\sin C \\ &= \frac{1}{2}ac\sin B \\ &= \frac{1}{2}bc\sin A. \end{aligned}$$

Theorem 1.8.4. (Heron's Formula) *For any triangle ABC,*

$$|\triangle ABC| = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a+b+c)$.

Proof. From our previous observation, we have that

$$|\triangle ABC| = \frac{1}{2}ab\sin C = \frac{1}{2}ab\sqrt{1-\cos^2 C}$$

and by the Law of Cosines $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$.

Thus,

$$
\begin{aligned}
|\triangle ABC| &= \frac{1}{2}ab\sqrt{1-\cos^2 C} \\
&= \frac{1}{4}\sqrt{4a^2b^2-(a^2+b^2-c^2)^2} \\
&= \frac{1}{4}\sqrt{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)} \\
&= \frac{1}{4}\sqrt{[(a+b)^2-c^2][c^2-(a-b)^2]} \\
&= \frac{1}{4}\sqrt{(a+b+c)(a+b+c-2c)(a+b+c-2b)(a+b+c-2a)} \\
&= \sqrt{\frac{(a+b+c)}{2}\left(\frac{(a+b+c)}{2}-c\right)\left(\frac{(a+b+c)}{2}-b\right)\left(\frac{(a+b+c)}{2}-a\right)} \\
&= \sqrt{s(s-c)(s-b)(s-a)}.
\end{aligned}
$$

$\square$

We now look at a special incircle.

The Pythagorean Inradius

Let $a = x^2 - y^2,\ b = 2xy,\ c = x^2 + y^2$ with $0 < y < x,\ \gcd(x, y) = 1$ and x and y being of opposite parity. Then (a, b, c) is a primitive Pythagorean triple. All Pythagorean triples can be generated by multiplying the above generating equations by λ, where λ ranges over all positive integers. Let triangle ABC, in the figure below, be a right triangle with integral sides a, b and integral hypotenuse c. Let the circle with center I be the inscribed circle for this triangle. We will prove that the inradius is an integer.

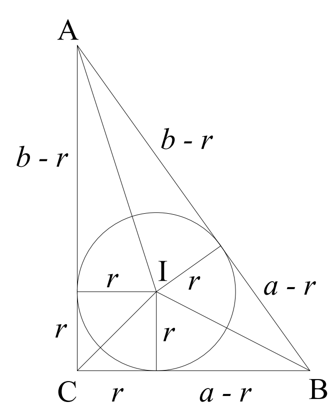


Figure 1.8.5

Proof. Let r be the inradius. Since the tangents to a circle from a point outside the circle are equal, we have the sides of triangle ABC configured as in Figure 1.8.5. Thus, $c = (a-r) + (b-r) = a + b - 2r$ and $r = \dfrac{a+b-c}{2}$. But $a = \lambda(x^2 - y^2),\ b = 2\lambda xy,$ and $c = \lambda(x^2 + y^2)$ which gives

$$r = \frac{\lambda(x^2 - y^2 + 2xy - x^2 - y^2)}{2} = \frac{\lambda(2xy - 2y^2)}{2} = \lambda y(x - y).$$

Thus, r is an integer given by $r = \lambda y(x - y)$.

An alternate proof uses the fact that

$$|\triangle ABC| = |\triangle AIB| + |\triangle BIC| + |\triangle AIC|.$$

So, $\frac{1}{2}ab = \frac{1}{2}rc + \frac{1}{2}ra + \frac{1}{2}rb$ and

$$\begin{aligned} r = \frac{ab}{a+b+c} &= \frac{\lambda^2(x^2 - y^2)2xy}{\lambda(x^2 - y^2 + 2xy + x^2 + y^2)} \\ &= \frac{\lambda(x^2 - y^2)2xy}{2x^2 + 2xy} \\ &= \frac{\lambda 2xy(x-y)(x+y)}{2x(x+y)} \\ &= \lambda y(x - y). \end{aligned}$$

Thus, r is an integer given by $r = \lambda y(x - y)$. $\square$

The Circumcircle and the Law of Sines

In addition to the incircle and excircles, there is another important circle associated with a triangle. This is the *circumcircle* and its center is the intersection of the perpendicular bisectors of the sides of the triangle. This center is called the *circumcenter* for the triangle and is usually denoted by O. The radius R of the circumcircle is called the *circumradius*. If in triangle ABC the midpoints of the sides BC, CA, AB are denoted by D, E, F, then the perpendiculars drawn at D, E, F meet in the circumcenter O and $OC = R$ is the radius of the circumcircle, as in Figure 1.8.6.

An important result in trigonometry is the Law of Sines which states: *For any triangle ABC with sides a, b, c, we have that* $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

This says that the ratio of the length of a side to the sine of the angle opposite that side is constant; however, in most trigonometry courses, we are not told what that common ratio is. As we see in the following theorem, the ratio is, indeed, a known quantity.

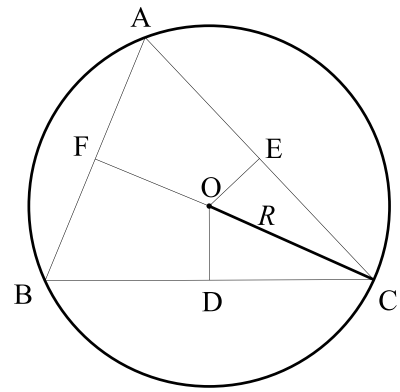


Figure 1.8.6

Theorem 1.8.5. (The Extended Law of Sines.) *For any triangle ABC with circumradius R*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Proof. Let triangle ABC with its circumscribed circle with center O and radius R be given. In the Figure 1.8.7, Figure A shows the case with all angles acute and Figure B shows the case with an obtuse angle. Draw diameter CD

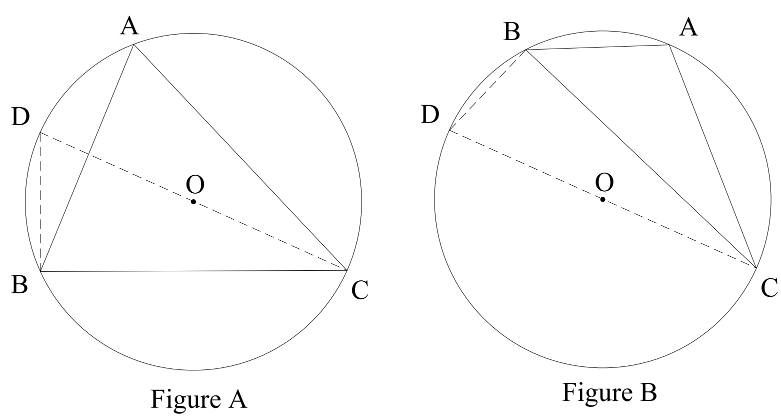


Figure 1.8.7: Law of Sines

and chord DB forming the right triangle CBD. $\angle CBD$ is a right angle since it is inscribed in a semicircle. In both figures we have that

$$\sin D = \frac{a}{CD} = \frac{a}{2R}.$$

In Figure A, $\angle D = \angle A$, since both angles have their vertex on the circle and subtend the same arc of the circle. However, in Figure B, $\angle D = 180^\circ - \angle A$, because they are opposite angles in a cyclic quadrilateral and such angles are supplementary. Since $\sin(180^\circ - \theta) = \sin\theta$, we have that $\sin D = \sin A$. Thus, regardless of whether $\angle A$ is acute or obtuse, we have that

$$\sin A = \frac{a}{2R}, \text{ or } \frac{a}{\sin A} = 2R.$$

We can apply the same procedure to the other angles of triangle ABC to obtain $\dfrac{b}{\sin B} = 2R$ and $\dfrac{c}{\sin C} = 2R$. Thus, we have the extended law of sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

□

The following problem is a classical problem that has many forms.

The Four Coin Problem. Suppose three congruent circles meet at a common point P and meet, in pairs, at the points A, B, and C, as in the figure below. Show that the circumcircle of triangle ABC has the same radius as the original circles.

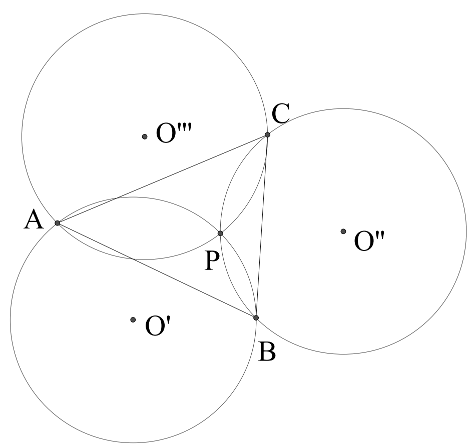


Figure 1.8.8: The Four Coin Problem

Solution. Suppose we look only at the points $A, B, C, O', O'', O''', P$. We then have Figure 1.8.9 with the solid lines forming three rhombuses (rhombii). All the solid lines are equal to the common radius, say r, of the three given circles. Furthermore, we have that $AO' \parallel O'''P \parallel CO''$, $AO''' \parallel O'P \parallel BO''$ and $O'B \parallel PO'' \parallel O'''C$. If we draw a dashed line through A parallel to $O'B$ and a dashed line through B parallel to $O'A$, meeting in a point we will label O, then we have

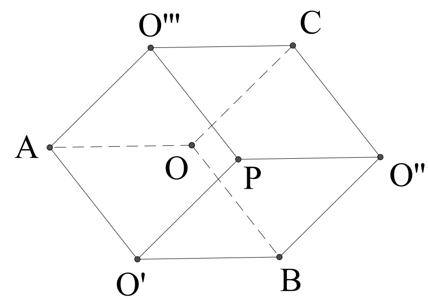


Figure 1.8.9

a parallelogram $AOBO'$ and since opposite sides of a parallelogram are equal, $|OA| = |BO'| = r$ and $|OB| = |AO'| = r$. We note further that OB is parallel to CO'' since it is parallel to AO' which is parallel to $O'''P$ which is, in turn, parallel to CO''. Since $|OB| = r = |CO''|$, the segment OC is parallel to BO''. Thus, $|OC| = |BO''| = r$ and the circle centered at O, with radius r, passes through A, B, and C, as we see in Figure 1.8.10. Figure 1.8.11 shows all the lines and circles. $\square$

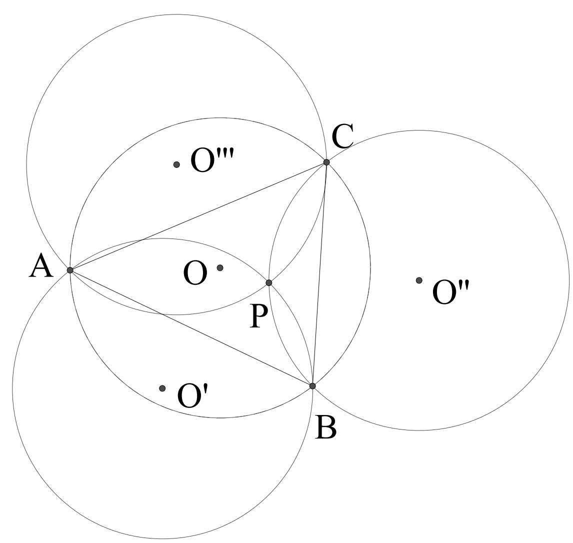


Figure 1.8.10

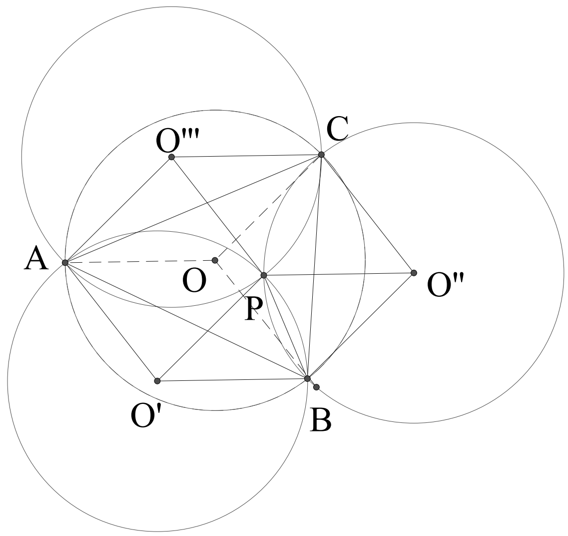


Figure 1.8.11: All Lines Drawn

Exercises

1.8.1. *Given a right triangle ABC with sides a, b and hypotenuse c. Use Heron's formula and the standard formula for the area of a triangle to prove the Pythagorean Theorem.*

1.8.2. *In* Figure 1.8.12 *the points $D, E, F, G, H, K, L, M, N$ are the contact points of the excircles to triangle ABC. If $s = \frac{1}{2}(a + b + c)$, prove the following*

(a) $BD = BF = s$

(b) $AG = AK = s$

(c) $CL = CN = s$

(d) $CF = CE = BL = BM = s - a$

(e) $AN = AM = CG = CH = s - b$

(f) $AD = AE = BK = BH = s - c$

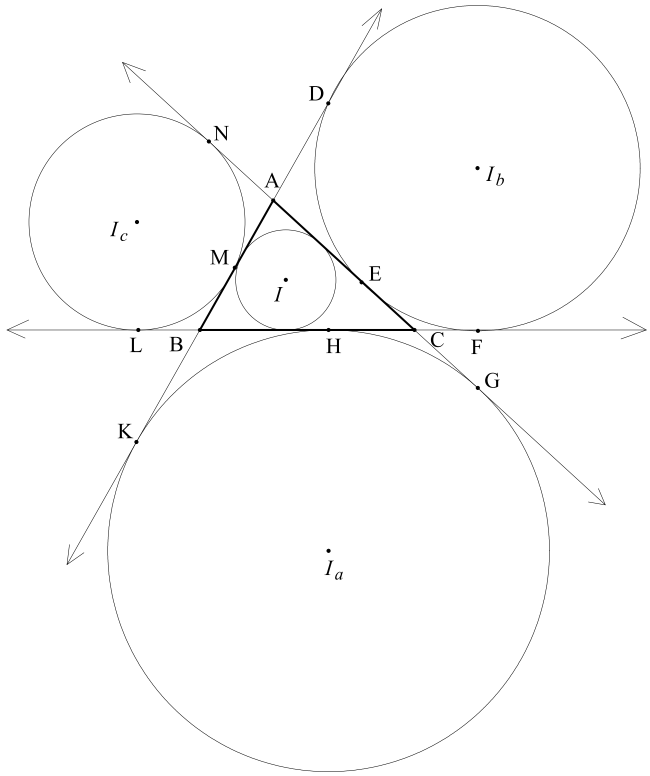


Figure 1.8.12: Excircles

Historical Note.

Heron of Alexandria (ca. 10 - 75) was a very talented mathematician, engineer and inventor. For centuries historians argued about when he lived ranging from 150 BC to 250 AD; however, in 1938 O. Neugebauer,in his book, *A history of ancient mathematical astronomy* (New York, 1975), discovered that Heron referred to a recent eclipse in one of his works. This eclipse took place in Alexandria on March 13, 62.

Heron was a prolific writer and inventor of many steam and water powered devices. He developed a steam engine that would propel a little vehicle, temple doors that appeared to open at a priests command, automated puppet shows, surveying devices and numerous objects of amusement. Some of his inventions can be seen at the URL:

http://www.mlahanas.de/Greeks/HeronAlexandria.htm

Chapter 2

Euclid
A Modern Perspective

2.1 Extending the Euclidean Plane

In order to facilitate our treatment of geometry we introduce the *extended plane.* In Euclidean geometry parallel lines never meet and, at times, this can be a problem. So a collection of points are added to the plane at infinity. These points are called *ideal points.* This then allows for any two lines in the plane to intersect in a point. Parallel lines will intersect in an ideal point. Each ideal point represents a direction in the plane, so that a family of parallel lines will all meet in the same ideal point. The set of all ideal points lie on a single line, called the *ideal line.*

The points and lines in the Euclidean plane are called *ordinary points* and *ordinary lines.* Triangles are said to be *ordinary triangles* if all vertices are ordinary points. Each ordinary line in the plane contains exactly one ideal point. For the extended plane we have the following theorem.

Theorem 2.1.1. *In the extended plane, any two distinct points determine one and only one line and any two distinct lines intersect in one and only one point.*

Historical Note.

The use of points at infinity was first used in a systematic way by Gérard Desargues (1593-1662). He is credited with inventing the geometry we call "projective geometry." He was very original in his work; however, his writing style made his work very difficult to read. This may be one of the reasons his work lay dormant until projective geometry was reinvented by Gaspard Monge(1746-1818). Projective geometry is considered to be a major advancement in the study of synthetic geometry. We will study a major theorem of Desargues' later in this chapter.

2.2 Sensed Magnitudes

Another innovation that improved the study of geometry is that of assigning a direction to movement on a line. We will find that assigning a direction to a line segment, called *sensed magnitudes*, will greatly enhance our ability to solve problems and prove theorems. The positive direction assigned to a line can be either direction. But once a direction is established if A and B are two points on the line, then the directed distance (or sensed magnitude) is defined to be such that $\overline{AB} = -\overline{BA}$. This could also be written as $\overline{AB} + \overline{BA} = 0$, which gives the obvious result that $\overline{AA} = 0$. The employment of direction greatly reduces the amount of writing and work that has to be done, as the following theorem illustrates.

Theorem 2.2.1. *If A, B, C are any three collinear points then*

$$\overline{AB} + \overline{BC} + \overline{CA} = 0.$$

Proof. If the points A, B, C are distinct, then C satisfies exactly one of the following conditions:

(i) C lies between A and B.

(ii) C lies on the prolongation of $\overline{AB}$.

(iii) C lies on the prolongation of $\overline{BA}$.

For Case (i) we have that $\overline{AB} = \overline{AC} + \overline{CB}$, or $\overline{AB} - \overline{AC} - \overline{CB} = 0$, which gives $\overline{AB} + \overline{BC} + \overline{CA} = 0$.
For Case (ii) we have that $\overline{AB} + \overline{BC} = \overline{AC}$, or $\overline{AB} + \overline{BC} - \overline{AC} = 0$, which gives $\overline{AB} + \overline{BC} + \overline{CA} = 0$.
For Case (iii) we have that $\overline{CA} + \overline{AB} = \overline{CB}$, or $\overline{CA} + \overline{AB} - \overline{CB} = 0$, which gives $\overline{AB} + \overline{BC} + \overline{CA} = 0$.

From the above proof, we see that there is economy in using directed distances. The equation $\overline{AB} + \overline{BC} + \overline{CA} = 0$ actually represents three statements in one.

Theorem 2.2.2. *Let O be any point on the line AB. Then $\overline{AB} = \overline{OB} - \overline{OA}$.*

Proof Exercise

Introducing a new point on a given line is often a useful tool in problem solving. We refer to such practice as inserting an origin on line AB. Sometimes, rather than inserting a new origin, we use an existing point on the given line, as we will see in the proof of the next theorem.

Theorem 2.2.3. (**Euler's Theorem**) *If A, B, C, D are any four collinear points, then*

$$\overline{AD} \cdot \overline{BC} + \overline{BD} \cdot \overline{CA} + \overline{CD} \cdot \overline{AB} = 0$$

Proof. Using the point D as an origin, we can write $\overline{BC} = \overline{DC} - \overline{DB}, \overline{CA} = \overline{DA} - \overline{DC}$, and $\overline{AB} = \overline{DB} - \overline{DA}$. We then have

$$\begin{aligned}
\overline{AD} \cdot \overline{BC} + \overline{BD} \cdot \overline{CA} &+ \overline{CD} \cdot \overline{AB} \\
&= \overline{AD}(\overline{DC} - \overline{DB}) + \overline{BD}(\overline{DA} - \overline{DC}) + \overline{CD}(\overline{DB} - \overline{DA}) \\
&= \overline{AD} \cdot \overline{DC} - \overline{AD} \cdot \overline{DB} + \overline{BD} \cdot \overline{DA} - \overline{BD} \cdot \overline{DC} \\
&\quad + \overline{CD} \cdot \overline{DB} - \overline{CD} \cdot \overline{DA} \\
&= 0
\end{aligned}$$

The first and sixth terms add to zero, since $-\overline{CD} \cdot \overline{DA} = -\overline{AD} \cdot \overline{DC}$. Likewise, the second and third terms add to zero, as do the fourth and fifth terms. Thus, we have the desired result

$$\overline{AD} \cdot \overline{BC} + \overline{BD} \cdot \overline{CA} + \overline{CD} \cdot \overline{AB} = 0$$

Theorem 2.2.4. (**Stewart's Theorem**) *If A, B, C are any three points on a line and P is any point, then*

$$\overline{PA}^2 \cdot \overline{BC} + \overline{PB}^2 \cdot \overline{CA} + \overline{PC}^2 \cdot \overline{AB} + \overline{BC} \cdot \overline{CA} \cdot \overline{AB} = 0.$$

Proof. We first note that it was not specified that P be on line AB. Therefore we must consider both cases: P on line AB, and P not on line AB. Using P as an origin on line AB, we have

$$\begin{aligned}
\overline{PA}^2 \cdot \overline{BC} &+ \overline{PB}^2 \cdot \overline{CA} + \overline{PC}^2 \cdot \overline{AB} + \overline{BC} \cdot \overline{CA} \cdot \overline{AB} \\
&= \overline{PA}^2(\overline{PC} - \overline{PB}) + \overline{PB}^2(\overline{PA} - \overline{PC}) + \overline{PC}^2(\overline{PB} - \overline{PA}) \\
&\quad + (\overline{PC} - \overline{PB})(\overline{PA} - \overline{PC})(\overline{PB} - \overline{PA}) \\
&= \overline{PA}^2 \cdot \overline{PC} - \overline{PA}^2 \cdot \overline{PB} + \overline{PB}^2 \cdot \overline{PA} - \overline{PB}^2 \cdot \overline{PC} + \overline{PC}^2 \cdot \overline{PB} \\
&\quad - \overline{PC}^2 \cdot PA + \overline{PC} \cdot \overline{PA} \cdot \overline{PB} - \overline{PC} \cdot \overline{PA}^2 - \overline{PC}^2 \cdot \overline{PB} \\
&\quad + \overline{PC}^2 \cdot \overline{PA} + \overline{PB} \cdot \overline{PA}^2 + \overline{PB}^2 \cdot \overline{PC} - \overline{PB} \cdot \overline{PC} \cdot \overline{PA} \\
&= 0
\end{aligned}$$

On the other hand, if P is not on line AB, let P' be the foot of the perpendicular from P to line AB. Let $PP' = h$. (See Figure 2.2.1)

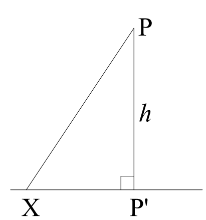


Figure 2.2.1

Then, for any point X on the line AB, we have $\overline{PX}^2 = \overline{P'X}^2 + h^2$. Thus, $\overline{PA}^2 = \overline{P'A}^2 + h^2$, $\overline{PB}^2 = \overline{P'B}^2 + h^2$, and $\overline{PC}^2 = \overline{P'C}^2 + h^2$. Substitution gives

$$\begin{aligned}
\overline{PA}^2 \cdot \overline{BC} &+ \overline{PB}^2 \cdot \overline{CA} + \overline{PC}^2 \cdot \overline{AB} + \overline{BC} \cdot \overline{CA} \cdot \overline{AB} \\
&= (\overline{P'A}^2 + h^2)\overline{BC} + (\overline{P'B}^2 + h^2)\overline{CA} + (\overline{P'C}^2 + h^2)\overline{AB} + \overline{BC} \cdot \overline{CA} \cdot \overline{AB} \\
&= \overline{P'A}^2 \cdot \overline{BC} + \overline{P'B}^2 \cdot \overline{CA} + \overline{P'C}^2 \cdot \overline{AB} + \overline{BC} \cdot \overline{CA} \cdot \overline{AB} + h^2(\overline{BC} + \overline{CA} + \overline{AB}) \\
&= 0
\end{aligned}$$

Since P' is on AB, $\overline{P'A}^2 \cdot \overline{BC} + \overline{P'B}^2 \cdot \overline{CA} + \overline{P'C}^2 \cdot \overline{AB} + \overline{BC} \cdot \overline{CA} \cdot \overline{AB} = 0$ and, by Theorem 2.2.1, $\overline{BC} + \overline{CA} + \overline{AB} = 0$.

Example 2.2.1. *If A, B, C are collinear points and a, b, c are the tangents from A, B, C to a given circle, then*

$$a^2\overline{BC} + b^2\overline{CA} + c^2\overline{AB} + \overline{BC} \cdot \overline{CA} \cdot \overline{AB} = 0.$$

Solution. Applying Stewart's Theorem with O, A, B, C in the figure below, we have

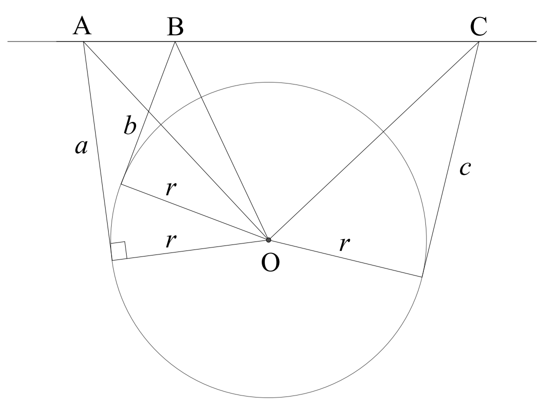


Figure 2.2.2

$$\overline{OA}^2 \cdot \overline{BC} + \overline{OB}^2 \cdot \overline{CA} + \overline{OC}^2 \cdot \overline{AB} + \overline{BC} \cdot \overline{CA} \cdot \overline{AB} = 0.$$

But $\overline{OA}^2 = a^2 + r^2$, $\overline{OB}^2 = b^2 + r^2$, and $\overline{OC}^2 = c^2 + r^2$. Thus,

$$(a^2 + r^2)\overline{BC} + (b^2 + r^2)\overline{CA} + (c^2 + r^2)\overline{AB} + \overline{BC} \cdot \overline{CA} \cdot \overline{AB} = 0.$$

Hence,

$$a^2\overline{BC} + b^2\overline{CA} + c^2\overline{AB} + \overline{BC} \cdot \overline{CA} \cdot \overline{AB} + r^2(\overline{BC} + \overline{CA} + \overline{AB}) = 0.$$

However, by Theorem 2.2.1, $\overline{BC} + \overline{CA} + \overline{AB} = 0$ and, therefore,

$$a^2\overline{BC} + b^2\overline{CA} + c^2\overline{AB} + \overline{BC} \cdot \overline{CA} \cdot \overline{AB} = 0.$$

Definition 2.2.1. *By convention, the angle $\angle AOB$ is generated by rotating the side OA about the point O until it coincides with OB, the rotation not exceeding 180°. If the rotation is counterclockwise, the angle is said to be* **positive***; if the rotation is clockwise, the angle is said to be* **negative**. *The directed angle will be denoted by $\angle\overline{AOB}$ with the condition that $\angle\overline{AOB} = -\angle\overline{BOA}$.*

Definition 2.2.2. *A triangle ABC will be considered as* **positive** *or* **negative** *according as the tracing of the perimeter from A to B to C to A is counterclockwise or clockwise. Such a signed triangular area is called a* **directed area** *and is denoted by $\triangle\overline{ABC}$.*

Theorem 2.2.5. *If vertex A of triangle ABC is joined to any point L on line BC, then*

$$\frac{\overline{BL}}{\overline{LC}} = \frac{AB \sin \overline{BAL}}{AC \sin \overline{LAC}}.$$

Proof. There are three possibilities for where L might lie, as we see in the figure below.

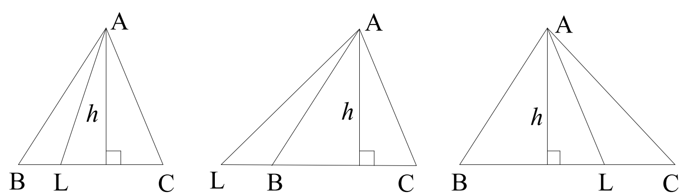


Figure 2.2.3

In all three cases we have that

$$\frac{\overline{BL}}{\overline{LC}} = \frac{h\overline{BL}}{h\overline{LC}} = \frac{2\triangle\overline{ABL}}{2\triangle\overline{ALC}} = \frac{(AB)(AL)\sin\overline{BAL}}{(AL)(AC)\sin\overline{LAC}} = \frac{AB\sin\overline{BAL}}{AC\sin\overline{LAC}}.$$

[**NOTE.** Recall that the area of a triangle can be computed as $\frac{1}{2}bc\sin A$, where b and c are any two sides of the triangle and A is the angle included between them.]

Exercises

2.2.1. *Prove Theorem* 2.2.2 .

2.2.2. *If AL is the bisector of angle A in triangle ABC, show that* $\dfrac{\overline{BL}}{\overline{LC}} = \dfrac{AB}{AC}$.

2.2.3. *If AL is the bisector of exterior angle A in triangle ABC, where $AB \neq AC$ show that* $\dfrac{\overline{BL}}{\overline{LC}} = -\dfrac{AB}{AC}$. [See Figure 2.2.4 on next page.]

2.2.4. *Find the lengths of the medians of a triangle having sides a, b, c.* Hint. Use Stewart's Theorem.

2.2.5. *Find the lengths of the angle bisectors of a triangle having sides a, b, c.* Hint. *Use Stewart's Theorem.*

2.2.6. *Prove the* Steiner-Lehmus Theorem*: If the bisectors of the base angles of a triangle are equal, the triangle is isosceles.*

2.2.7. *Show that the sum of the squares of the distances of the vertex of the right angle of a right triangle from the two points of trisection of the hypotenuse is equal to $\frac{5}{9}$ the square of the hypotenuse.*

2.2.8. *If A, B, P are collinear and M is the midpoint of AB, show that* $\overline{PM} = \dfrac{\overline{PA} + \overline{PB}}{2}$.

2.2.9. *If O, A, B, C are collinear and $\overline{OA} + \overline{OB} + \overline{OC} = 0$ and if P is any point on line AB, show that $\overline{PA} + \overline{PB} + \overline{PC} = 3\overline{PO}$.*

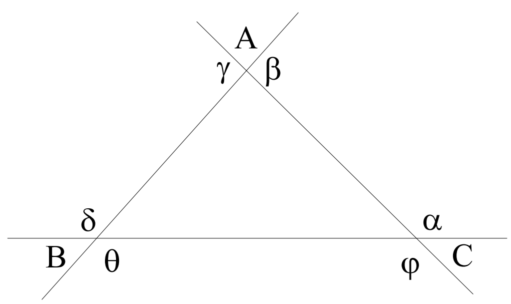


Figure 2.2.4: The angles $\alpha, \beta, \gamma, \delta, \theta$ and φ are exterior angles of $\triangle ABC$

2.3 Menelaus' and Ceva's Theorems

Very little is written about Menelaus of Alexandria (c.a. 100 CE); however, Ptolemy and Pappus both refer to his works, of which there were many although none are extant. One book, in particular, on spherical triangles, *Sphaerica*, which is preserved through its Arabic translation, contains the theorem named for him. As is often the case in mathematics, the theorem for plane triangles was known before Menelaus wrote his work on spherical triangles. The theorem for plane triangles was stated as follows:

> *If a straight line crosses the three sides of a triangle (one of the sides is extended beyond the vertices of the triangle), then the product of three of the nonadjacent line segments thus formed is equal to the product of the three remaining line segments of the triangle.*

In more modern times the concept of directed distance is used to give better understanding of what the theorem implies. For clarity we make the following definition of a menelaus point.

Definition 2.3.1. *An ordinary or ideal point lying on a side line of an ordinary triangle, but not coinciding with a vertex of the triangle will be called a* **menelaus point** *of the triangle for this side.*

We can now examine the modern version of Menelaus' theorem.

Theorem 2.3.1. **[Menelaus' Theorem]** *A necessary and sufficient condition for three menelaus points D, E, F for the sides BC, CA, AB of an ordinary triangle ABC to be collinear is that*

$$\frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = -1.$$

Proof. *Necessity.* In Figure 2.3.1 we let ℓ be the line containing the three menelaus points D, E, F and we draw the line CG from C parallel to the side BA of the triangle ABC.

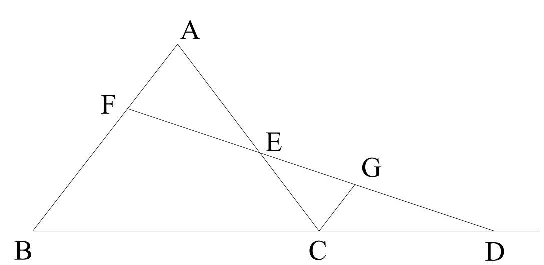


Figure 2.3.1

In the resulting figure $\triangle BFD \sim \triangle CGD$ and $\triangle AFE \sim \triangle CGE$. In the first case we have, disregarding signs, $\frac{BD}{CD} = \frac{BF}{CG}$, which can be written

$$\frac{BD}{CD \cdot BF} = \frac{1}{CG}. \tag{2.1}$$

In the second case we have $\frac{AF}{CG} = \frac{AE}{CE}$, which can be written

$$\frac{AF \cdot CE}{AE} = CG. \tag{2.2}$$

Multiplying (2.1) and (2.2) together and rearranging terms, we get

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{CG}{CG} = 1.$$

Now, assigning direction and observing that ℓ must cut either one or all three of the sides externally, we have

$$\frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = -1.$$

If ℓ is the line at infinity, then each of the ratios would be -1 and the result follows.

Sufficiency. Suppose

$$\frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = -1. \tag{2.3}$$

Let us further suppose that EF cuts BC in a point D^*. Then D^* is a menelaus point and by the necessity part of the theorem it follows that

$$\frac{\overline{BD^*}}{\overline{D^*C}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = -1. \tag{2.4}$$

Now from equations (2.3) and (2.4) it follows that

$$\frac{\overline{BD^*}}{\overline{D^*C}} = \frac{\overline{BD}}{\overline{DC}}$$

and $D^* = D$. That is, D, E, F are collinear.

We now turn to a similar theorem for concurrent lines discovered by Giovanni Ceva.

Giovanni Ceva (1647-1734) was born in Milan. He later studied at the university of Pisa. He taught at Pisa before being appointed Professor of mathematics at the University of Mantua in 1686, where he remained for the rest of his life. Most of Ceva's work was in the field of geometry. In addition to rediscovering and publishing Menelaus' Theorem, he discovered what many consider to be one of the most important results on the synthetic geometry of the triangle between Greek times and the 19th Century. The theorem states that

> *... lines from the vertices of a triangle to the opposite sides are concurrent precisely when the product of the ratio the sides are divided is 1.*

Definition 2.3.2. *A line passing through a vertex of an ordinary triangle, but not coinciding with a side of the triangle, will be called a* **cevian line**.

Theorem 2.3.2. [**Ceva's Theorem**] *A necessary and sufficient condition for three cevian lines AD, BE, CF of an ordinary triangle ABC to be concurrent is that*

$$\frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = 1.$$

Proof. *Necessity.* In Figure 2.3.2, suppose AD, BE, CF are concurrent in P. It may be assumed, without loss of generality, that P does not lie on the line through A, parallel to BC.

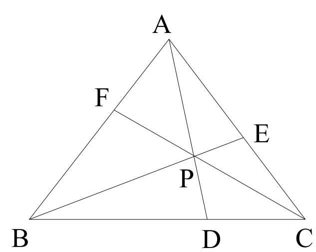


Figure 2.3.2

Consider triangle ABD with transversal FPC. Applying Menelaus' theorem we get

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BC}}{\overline{CD}} \cdot \frac{\overline{DP}}{\overline{PA}} = -1 \tag{2.5}$$

Next, using triangle ADC and transversal BPE we get

$$\frac{\overline{DB}}{\overline{BC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AP}}{\overline{PD}} = -1 \tag{2.6}$$

Multiplying equations (2.5) and (2.6) together, we obtain

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BC}}{\overline{CD}} \cdot \frac{\overline{DP}}{\overline{PA}} \cdot \frac{\overline{DB}}{\overline{BC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AP}}{\overline{PD}} = 1 \tag{2.7}$$

Noting that $\overline{DB} = -\overline{BD}$, $\overline{CD} = -\overline{DC}$, $\overline{PA} = -\overline{AP}$ and $\overline{DP} = -\overline{PD}$, we can cancel like factors and rearrange to get

$$\frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = 1.$$

Sufficiency. Suppose

$$\frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = 1 \tag{2.8}$$

and let BE, CF intersect in P. Draw AP to cut BC in D^*. Then AD^* is a cevian line. Hence, by the necessity part of this theorem, we have

$$\frac{\overline{BD^*}}{\overline{D^*C}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = 1. \tag{2.9}$$

Now from equations (2.8) and (2.9) it follows that

$$\frac{\overline{BD^*}}{\overline{D^*C}} = \frac{\overline{BD}}{\overline{DC}}$$

and $D^* = D$. That is, AD, BE, CF are concurrent.

Both Menelaus' and Ceva's theorems have trigonometric forms. They depend on Theorem 2.2.5.

Theorem 2.3.3. (**Trigonometric Form of Menelaus' Theorem**) *A necessary and sufficient condition for three menelaus points D, E, F for the sides BC, CA, AB of an ordinary triangle ABC to be collinear is that*

$$\frac{\sin \overline{BAD}}{\sin \overline{DAC}} \cdot \frac{\sin \overline{CBE}}{\sin \overline{EBA}} \cdot \frac{\sin \overline{ACF}}{\sin \overline{FCB}} = -1.$$

Proof. We use the figure below where dashed lines are added to better see the angles referred to in the theorem and proof.

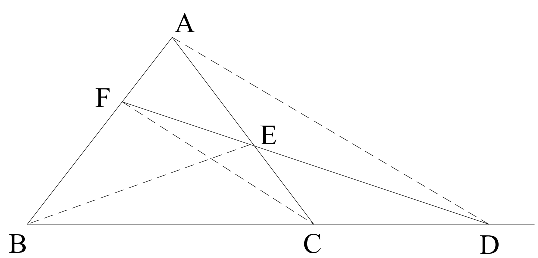


Figure 2.3.3

From Theorem 2.2.5 we have the following on the three sides of the triangle.

$$\begin{aligned}
\frac{\overline{BD}}{\overline{DC}} &= \frac{AB \sin \overline{BAD}}{AC \sin \overline{DAC}} \\
\frac{\overline{CE}}{\overline{EA}} &= \frac{BC \sin \overline{CBE}}{BA \sin \overline{EBA}} \\
\frac{\overline{AF}}{\overline{FB}} &= \frac{CA \sin \overline{ACF}}{CB \sin \overline{FCB}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\overline{BD}}{\overline{DC}}\cdot\frac{\overline{CE}}{\overline{EA}}\cdot\frac{\overline{AF}}{\overline{FB}} &= \frac{AB\sin\overline{BAD}}{AC\sin\overline{DAC}}\,\frac{BC\sin\overline{CBE}}{BA\sin\overline{EBA}}\,\frac{CA\sin\overline{ACF}}{CB\sin\overline{FCB}}\\
&= \frac{AB\cdot BC\cdot CA\cdot\sin\overline{BAD}\sin\overline{CBE}\sin\overline{ACF}}{AC\cdot BA\cdot CB\cdot\sin\overline{DAC}\sin\overline{EBA}\sin\overline{FCB}}\\
&= \frac{\sin\overline{BAD}\sin\overline{CBE}\sin\overline{ACF}}{\sin\overline{DAC}\sin\overline{EBA}\sin\overline{FCB}}
\end{aligned}$$

Therefore it follows that

$$\frac{\overline{BD}}{\overline{DC}}\cdot\frac{\overline{CE}}{\overline{EA}}\cdot\frac{\overline{AF}}{\overline{FB}} = -1 \text{ if and only if } \frac{\sin\overline{BAD}\sin\overline{CBE}\sin\overline{ACF}}{\sin\overline{DAC}\sin\overline{EBA}\sin\overline{FCB}} = -1.$$

Theorem 2.3.4. (**Trigonometric Form of Ceva's Theorem**) *A necessary and sufficient condition for three cevian lines AD, BE, CF of an ordinary triangle ABC to be concurrent is that*

$$\frac{\sin\overline{BAD}}{\sin\overline{DAC}}\cdot\frac{\sin\overline{CBE}}{\sin\overline{EBA}}\cdot\frac{\sin\overline{ACF}}{\sin\overline{FCB}} = 1.$$

Proof. **Exercise**

We now introduce two nice applications of the theorems of Menelaus and Ceva.

Theorem 2.3.5. *If AD, BE, CF are any three concurrent cevian lines of an ordinary triangle ABC, and if D' denotes the point of intersection of BC and FE, then D and D' divide BC, one (D) internally and the other (D') externally in the same numerical ratio.*

Proof. Consider the figure below. We are given that AD, BE, CF are any

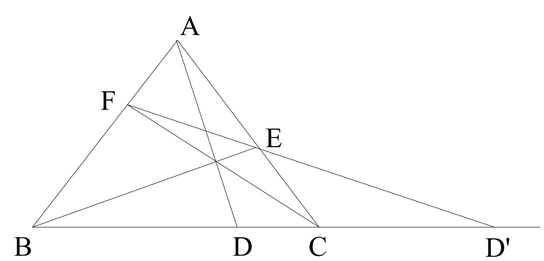


Figure 2.3.4

three concurrent cevian lines, so we have by Ceva's theorem

$$\frac{\overline{BD}}{\overline{DC}}\cdot\frac{\overline{CE}}{\overline{EA}}\cdot\frac{\overline{AF}}{\overline{FB}} = 1.$$

Furthermore, we have that D', E, F are three collinear menelaus points. Hence, by Menelaus' theorem

$$\frac{\overline{BD'}}{\overline{D'C}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = -1.$$

Equating these, we get

$$\frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = -\frac{\overline{BD'}}{\overline{D'C}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}}$$

which reduces to

$$\frac{\overline{BD}}{\overline{DC}} = -\frac{\overline{BD'}}{\overline{D'C}}.$$

Therefore, D and D' divide BC, one internally and one externally, in the same numerical ratio.

From the above theorem we have the following interesting result.

> Given a segment BC and a point D (or D') that divides the segment BC internally (externally) in some numerical ratio, we can construct the other division point.

What is very interesting is that this construction can be done with straightedge only!

Construction 2.3.1. *Given a line segment BC and a point D, which divides BC internally in the ratio $\frac{\overline{BD}}{\overline{DC}}$. Construct D' so that $\frac{\overline{BD'}}{\overline{D'C}} = -\frac{\overline{BD}}{\overline{DC}}$.*

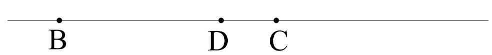


Figure 2.3.5

Solution. In Figure 2.3.5, pick any point A not on line BC. Join A with each of the points B, C, D.

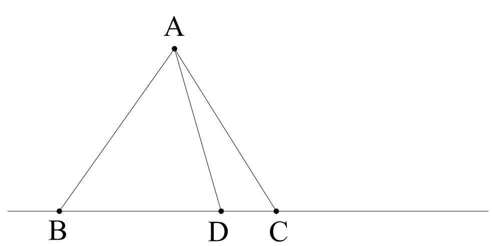


Figure 2.3.6

Next draw any line from B cutting AC in a point E and AD in point P. Draw CP extended cutting AB in the point F. Draw FE extended to cut the

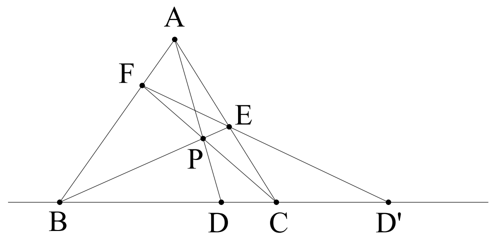


Figure 2.3.7

extension of BC in the desired point D'

NOTE. The position of A is not important. Furthermore, it should be noted that if D is close to the midpoint of segment BC, D' will approach an ideal point.

Construction 2.3.2. *Given a segment BC and a point D on BC extended which divides BC externally in the ratio $\frac{\overline{BD'}}{\overline{D'C}}$. Construct a point D such that $\frac{\overline{BD}}{\overline{DC}} = -\frac{\overline{BD'}}{\overline{D'C}}$*

Figure 2.3.8

Solution. Pick any point A, not on line BC and connect A with B and C. Then draw a line from D meeting AC at E and AB at F.

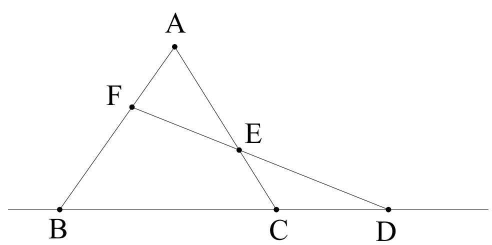


Figure 2.3.9

Draw BE and CF meeting in P. Draw AP intersecting BC at D' the desired point. See Figure 2.3.10

NOTE. Theorem 2.3.5 establishes the validity of these two constructions.

The next theorem is a very important result in projective geometry.

Definition 2.3.3. *Two triangles ABC and $A'B'C'$ are said to be* **copolar** *if AA', BB', CC' are concurrent.*

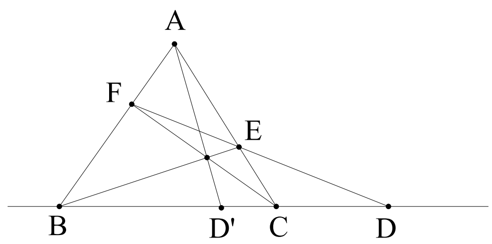


Figure 2.3.10

In Figure 2.3.11 below ABC and $A'B'C'$ are copolar from the point O.

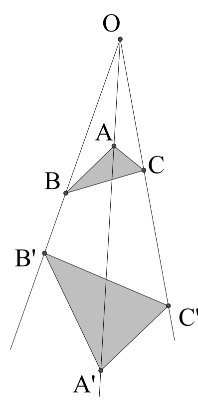


Figure 2.3.11: Copolar Triangles

Definition 2.3.4. *Two triangles ABC and $A'B'C'$ are said to be* **coaxial** *if the points of intersection of BC and $B'C'$, CA and $C'A'$, AB and $A'B'$ are collinear.*

In Figure 2.3.12 below ABC and $A'B'C'$ are coaxial from the line containing the points $P,\ Q,\ R$.

NOTE. Copolar triangles are also said to be *perspective from a point* and coaxial triangles are said to be *perspective from a line.*

Theorem 2.3.6. (**Desargues' Theorem**) *Copolar triangles are coaxial, and conversely.*

Proof. Let the two triangles be ABC and $A'B'C'$ in Figure 2.3.13 below. Suppose $AA',\ BB'\ CC'$ are concurrent at point O. Let $P,\ Q,\ R$ be the points

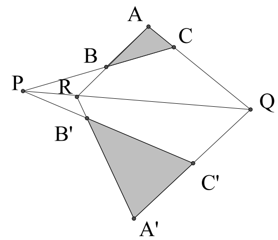


Figure 2.3.12: Coaxial Triangles

of intersection of BC and $B'C'$, AC and $A'C'$, AB and $A'B'$, respectively.

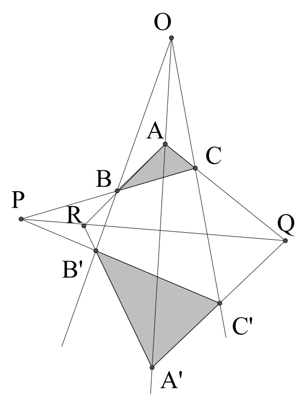


Figure 2.3.13: Desargues' Theorem

Using triangle BCO with transversal (a line cutting all three sides) $B'C'P$, we have by Menelaus' theorem,

$$\frac{\overline{BP}}{\overline{PC}} \cdot \frac{\overline{CC'}}{\overline{C'O}} \cdot \frac{\overline{OB'}}{\overline{B'B}} = -1 \tag{2.10}$$

Now with triangle CAO with transversal $C'A'Q$ we have by Menelaus' theorem,

$$\frac{\overline{CQ}}{\overline{QA}} \cdot \frac{\overline{AA'}}{\overline{A'O}} \cdot \frac{\overline{OC'}}{\overline{C'C}} = -1 \tag{2.11}$$

And, finally, using triangle ABO with transversal $A'B'R$ we have by Menelaus' theorem,

$$\frac{\overline{AR}}{\overline{RB}} \cdot \frac{\overline{BB'}}{\overline{B'O}} \cdot \frac{\overline{OA'}}{\overline{A'A}} = -1 \tag{2.12}$$

Multiplying (2.10), (2.11) and (2.12) we have

$$\frac{\overline{BP}}{\overline{PC}} \cdot \frac{\overline{CC'}}{\overline{C'O}} \cdot \frac{\overline{OB'}}{\overline{B'B}} \cdot \frac{\overline{CQ}}{\overline{QA}} \cdot \frac{\overline{AA'}}{\overline{A'O}} \cdot \frac{\overline{OC'}}{\overline{C'C}} \cdot \frac{\overline{AR}}{\overline{RB}} \cdot \frac{\overline{BB'}}{\overline{B'O}} \cdot \frac{\overline{OA'}}{\overline{A'A}} = -1$$

Now, in the above equation, we find six factors in each numerator and denominator that match up except for being oppositely directed. Therefore they will cancel out as $(-1)^6 = 1$. Therefore, we have after the reduction,

$$\frac{\overline{BP}}{\overline{PC}} \cdot \frac{\overline{CQ}}{\overline{QA}} \cdot \frac{\overline{AR}}{\overline{RB}} = -1$$

and, by Menelaus' theorem, P, Q, R are collinear. Thus copolar triangles are coaxial.

Conversely, suppose the triangles are coaxial; that is, suppose P, Q, R are collinear and let O be the point of intersection of BB' and CC'. Now triangles CQC' and BRB' are copolar from P and therefore, by the first part of Desargues' theorem, coaxial from O, A, A'. That is, O, A, A' are collinear and AA' meets BB' and CC' in O. Thus coaxial triangles are copolar.

Exercises

2.3.1. *Prove that the medians of a triangle are concurrent. (This point is called the centroid of the triangle.)*

2.3.2. *Prove that the altitudes of a triangle are concurrent. (This point is called the orthocenter of the triangle.)*

2.3.3. *Prove that if two triangles share a common base and a common cevian line joining the nonshared vertices, then the ratio of the lengths of the common cevian line is equal to the ratio of the areas of the two triangles.*

2.3.4. *If D, E, F are the points of contact of the inscribed circle of triangle ABC with sides BC, CA, AB respectively, show that the lines AD, BE, CF are concurrent. This point is called the* ***Gergonne point of the triangle.***

2.3.5. *Let D, E, F be the points on the sides BC, CA, AB of triangle ABC such that D is half way around the perimeter from A, E is half way around the perimeter from B, and F is half way around the perimeter from C. Show that AD, BE, CF are concurrent. This point is called the* ***Nagel point of the triangle****.*

2.3.6. *Prove the trigonometric form of Ceva's Theorem.*

Definition 2.3.5. *Let X and X' be points on a line segment PQ which are symmetric with respect to the midpoint of PQ. Then X and X' are called a pair of* ***isotomic points*** *for the segment PQ.* See Figure 2.3.14.

Figure 2.3.14

2.3.7. *Show that if D and D', E and E', F and F', are isotomic points for the sides of triangle ABC, and if AD, BE, CF are concurrent, then AD', BE', CF' are also concurrent. (Two such related points of concurrency are called a pair of isotomic conjugate points for a triangle.)*

2.3.8. *Let A, B, C, D, M be collinear with M the midpoint of AB and C between A and M and D between M and B. Show that if $\overline{AD} = \overline{CB}$, then $\overline{CM} = \overline{MD}$.*

2.3.9. *Show that the Gergonne and Nagel points of a triangle are a pair of isotomic conjugate points of the triangle.*

2.3.10. *Show that the tangents to the circumcircle of a triangle at the vertices of the triangle intersect the opposite sides of the triangle in three collinear points.*

2.3.11. *If AD, BE, CF are three cevian lines of an ordinary triangle ABC, concurrent at a point P, and if EF, DF and ED intersect the sides BC, CA and AB of triangle ABC in the points D', E', F', show that D', E', F' are collinear.*

2.3.12. *If equilateral triangles BCA', CAB', ABC' are described externally upon the sides BC, CA, AB of triangle ABC, show that AA', BB', CC' are concurrent in a point.*

2.3.13. *Prove that the external bisectors of the angles of a triangle intersect the opposite sides in three collinear points.*

2.3.14. *Given a triangle ABC, as in Figure 2.3.15, with EF parallel to BC. Let P be the intersection of BE and CF and let AP meet BC in D and EF in G. Prove that $BD = DC$ and $FG = GE$.* (Hint. Use Ceva's theorem and Theorem 1.3.5.)

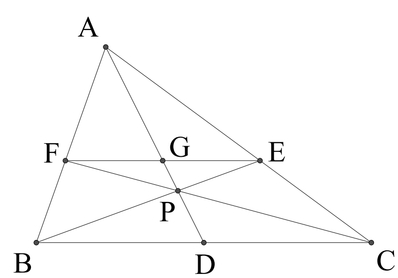


Figure 2.3.15

2.3.15. *Place 10 points and 10 lines in a plane such that each line contains exactly three of the points and exactly three lines concur at each of the points.*

2.3.16. *Let I_c be the center of an excircle for triangle ABC. Let D, E, F be the points of tangency to this circle with sides BC, CA, AB, respectively. Prove that AD, BE, CF are concurrent.*

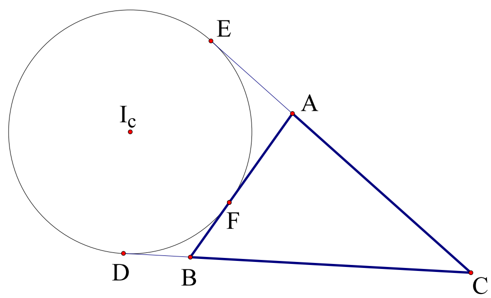


Figure 2.3.16

2.4 A Projection Problem

This material is from an article of mine that appeared in the *American Mathematical Monthly*, Vol. 73, No. 6, June-July, 1966.

As an undergraduate student in a projective geometry course, I was assigned the following problem:

Show that any quadrangle may be projected into a parallelogram.

The solution of this problem usually went as follows: Let $PQRS$ be the given quadrangle with exterior diagonal points T and U. Let O be any point not in the plane of $PQRS$ and join the points P, Q, R, S, T, U with O. See Figure 2.4.1. Now if the set of lines OP, OQ, OR, OS are cut by a plane α parallel to the plane OTU we get points P' on OP, Q' on OQ, R' on OR and S' on OS, which when joined form a parallelogram. The fact that $P'Q'R'S'$ is a parallelogram is established as follows: T is on the line of intersection of the planes of OUT and $PQRS$, therefore its image T' on α must be an ideal point since α and the plane OUT are parallel. Now T' would be the intersection of $P'Q'$ and $R'S'$, so it follows that $P'Q'$ and $R'S'$ are parallel. A similar argument establishes that $P'S'$ and $Q'R'$ are parallel.

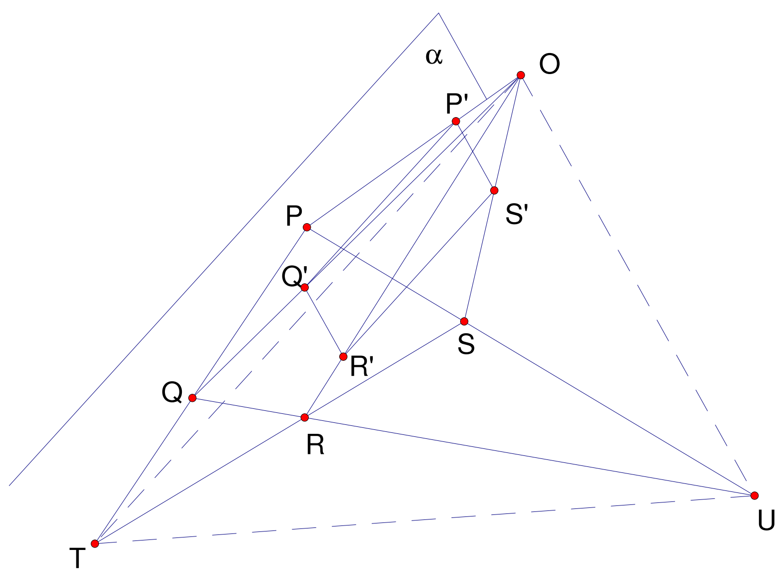


Figure 2.4.1: O not in the plane of $PQRS$

What intrigued me about this problem was the fact that although we were dealing with a three dimensional solution to the problem, the two dimensional drawing in Figure 2.4.1 made the problem appear solvable in the plane. This led to the investigation of the case in which the point O is in the plane of $PQRS$.

More precisely, given a quadrangle $PQRS$ and a point O in its plane, construct a line X_1X_2 such that there is a parallelogram $P'Q'R'S'$ which is copolar with $PQRS$ from the point O and coaxial from the line X_1X_2. See Figure 2.4.3.

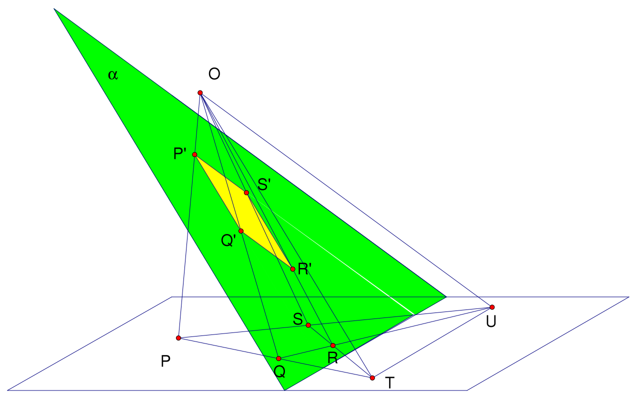


Figure 2.4.2: Alternate View of O not in the plane of $PQRS$

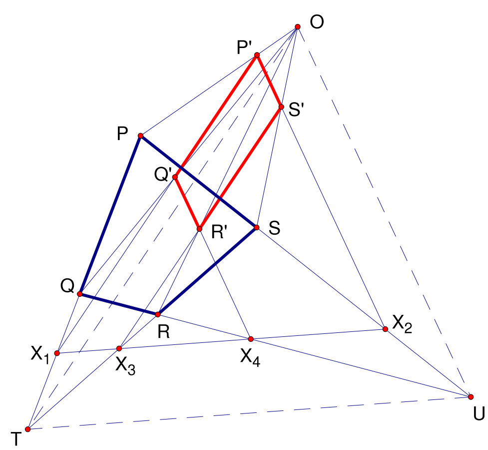


Figure 2.4.3: O in the plane of $PQRS$

If O is in the plane of $PQRS$, the construction is as follows. Join O with the vertices P, Q, R, S and the exterior diagonal points T and U. On the line OP pick a point P' and construct lines $P'X_1$ and $P'X_2$ parallel to OT and OU, respectively. $P'X_1$ meets OQ in a point Q' and $P'X_2$ meets OS in a point S'. Join the points X_1 and X_2. The line X_1X_2 meets ST at X_3 and QU at X_4.

Join the points X_3 and S' which meet OR at R'. Join Q' and X_4. This line intersects OR at R' and the figure $P'Q'R'S'$ is the required parallelogram.

Proof. Triangles $OP'Q'$ and UX_2X_4 are coaxial from the line PQ since OP' meets UX_2 in P, OQ' meets UX_4 in Q and $P'Q'$ meets X_2X_4 in X_1. By Desargues' Theorem, these triangles are copolar. By construction, $P'X_2$ is parallel to OU. Therefore, the pole is an ideal point, and thus $Q'X_4$ is parallel to $P'X_2$. Since triangles $OP'S'$ and TX_1X_3 are coaxial from line PS, we see, by a similar argument, that $S'X_3$ is parallel to $P'X_1$. All that remains to prove is that $Q'X_4, S'X_3$ and OR are concurrent at R'.

Suppose $S'X_3$ meets OR at R'. Triangles $OR'S'$ and UX_4X_2 are coaxial from the line RS, and therefore they must be copolar. Line $S'X_2$ was constructed parallel to OU; therefore $R'X_4$ is parallel to OU. On the other hand, it has already been established that $Q'X_4$ is parallel to OU. Since one and only one line can be drawn parallel to a given line through a given point, R' is on line $Q'X_4$. Therefore, lines $OR, Q'X_4, S'X_3$ are concurrent on R'.

It is worth noting that if O is chosen to be on the circle with diameter TU, the parallelogram $P'Q'R'S'$ will be a rectangle.

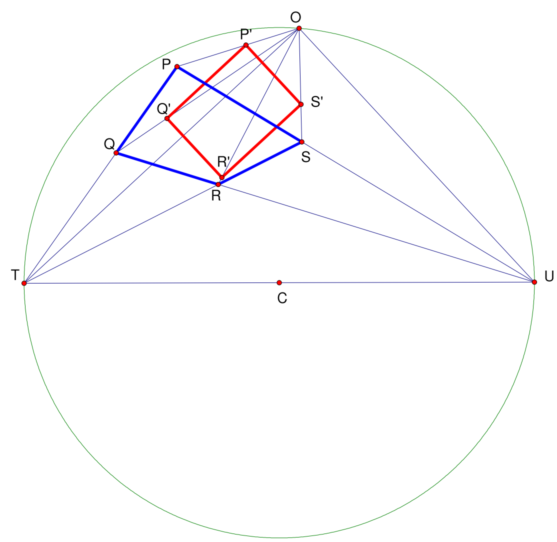


Figure 2.4.4: O on circle with diameter TU

Note. As O moves along the semicircle from T to U, the rectangles go from ones for which $P'Q' < P'S'$ to ones for which $P'Q' > P'S'$. Therefore, it would appear that somewhere along the semicircle there is a locus for O such that $P'Q' = P'S'$ and $P'Q'R'S'$ is a square.

2.5 Cross Ratio

The cross ratio is also referred to as an *anharmonic ratio* or as a *double ratio.* The history of the cross ratio dates back to the ancient Greeks; however, the more modern idea of sensed magnitudes has allowed for a nice notation and convenience of manipulation. The modern treatment of cross ratios dates back to the early 19th century and the work of Möbius. It was Möbius who introduced the notation we will be using.

Definition 2.5.1. *If A, B, C, D are four distinct points on an ordinary line, we use the symbol (AB, CD) to represent the ratio of ratios $(\overline{AC}/\overline{CB})/(\overline{AD}/\overline{DB})$ and call it the* **cross ratio** *of the range of points A, B, C, D taken in this order.*

Note. Regardless of the order of the points on the line, the cross ratio (AB, CD) is computed as $(\overline{AC}/\overline{CB})/(\overline{AD}/\overline{DB})$.

Example 2.5.1. *Let the points A, B, C, D be points on the number line given by*

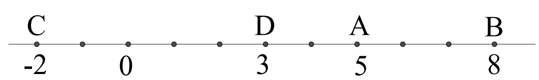


Figure 2.5.1

To compute (AB, CD) we find $\overline{AC} = -7$, $\overline{CB} = 10$, $\overline{AD} = -2$ and $\overline{DB} = 5$. Therefore, $(AB, CD) = (\overline{AC}/\overline{CB})/(\overline{AD}/\overline{DB}) = (-7/10)/(-2/5) = 7/4$

If we want to compute (CB, AD), we need to compute it as $(CB, AD) = (\overline{CA}/\overline{AB}))/(\overline{CD}/\overline{DB}) = (7/3)/(5/5) = 7/3$.

The order of the points within the cross ratio is critical in how the cross ratio is computed. Since there are twenty-four permutations of four points, it appears we may have a problem. However, it is not as bad as it seems. As it turns out, there are only six distinct values for the twenty-four rearrangements. That is, we can partition the twenty-four arrangements of the four points into six classes, with every element in each class having the same value. We begin by looking at three basic methods of rearrangement.

Theorem 2.5.1. *If we let $(AB, CD) = r$ and in the symbol (AB, CD) we*

1. *interchange any two of the points and at the same time interchange the other two points, the cross ratio's value is unaltered,*

2. *we interchange only the first pair of points, the resulting cross ratio has value* $\frac{1}{r}$

3. *we interchange only the middle pair of points, the resulting cross ratio has value* $1 - r$.

Before we prove this theorem, let us examine its implications. For example, Part (1) gives

$$(AB, CD) = (BA, DC) = (CD, AB) = (DC, BA) = r.$$

To get (BA, DC) we interchanged A and B and at the same time C and D. To get (CD, AB) we switched A and C and at the same time B and D. To get (DC, BA) we switched A and D and at the same time B and C.

For Part (2) we interchange only the first pair obtaining $(BA, CD) = \frac{1}{r}$. But then we can apply the rearrangements given in Part (1) to get

$$(BA, CD) = (AB, DC) = (DC, AB) = (CD, BA) = \frac{1}{r}.$$

Part (3) says that $(AC, BD) = 1 - r$. Again, applying the rearrangements of Part (1) to this cross ratio gives

$$(AC, BD) = (BD, AC) = (CA, DB) = (DB, CA) = 1 - r.$$

The proof for Theorem 2.5.1 is rather straightforward. From $(AB, CD) = r$, we can get the results of Part (1) by expanding and using the properties of ratios.

$$(BA, DC) = (\overline{BD}/\overline{DA})/(\overline{BC}/\overline{CA}) = \frac{\overline{BD} \cdot \overline{CA}}{\overline{DA} \cdot \overline{BC}} = \frac{\overline{AC} \cdot \overline{DB}}{\overline{CB} \cdot \overline{AD}} = (AB, CD).$$

The remaining two are left as an exercise.

For Part (2), we get by expansion

$$(BA, CD) = \frac{\overline{BC}}{\overline{CA}} \Bigg/ \frac{\overline{BD}}{\overline{DA}} = \frac{\overline{BC} \cdot \overline{DA}}{\overline{CA} \cdot \overline{BD}} = \frac{1}{\frac{\overline{AC} \cdot \overline{DB}}{\overline{CB} \cdot \overline{AD}}} = \frac{1}{r}.$$

The others in Part (2) follow from Part (1).

Part (3) requires a little more effort. We wish to show that $(AC, BD) = 1 - r$.

Expanding the left side gives,

$$
\begin{aligned}
(AC, BD) &= \left(\frac{\overline{AB}}{\overline{BC}}\right) \bigg/ \left(\frac{\overline{AD}}{\overline{DC}}\right) \\
&= \left(\frac{\overline{AC} + \overline{CB}}{\overline{BC}}\right) \bigg/ \left(\frac{\overline{AD}}{\overline{DB} + \overline{BC}}\right) \\
&= \frac{(\overline{AC} + \overline{CB})(\overline{DB} + \overline{BC})}{\overline{BC} \cdot \overline{AD}} \\
&= \frac{\overline{AC} \cdot \overline{DB} + \overline{CB} \cdot \overline{DB} + \overline{AC} \cdot \overline{BC} + \overline{CB} \cdot \overline{BC}}{\overline{BC} \cdot \overline{AD}} \\
&= \frac{\overline{AC} \cdot \overline{DB}}{\overline{BC} \cdot \overline{AD}} + \frac{\overline{CB} \cdot \overline{DB} + \overline{AC} \cdot \overline{BC} + \overline{CB} \cdot \overline{BC}}{\overline{BC} \cdot \overline{AD}} \\
&= -\frac{\overline{AC} \cdot \overline{DB}}{\overline{CB} \cdot \overline{AD}} + \frac{\overline{BC}(\overline{BD} + \overline{AC} + \overline{CB})}{\overline{BC} \cdot \overline{AD}} \\
&= -r + \frac{\overline{AD}}{\overline{AD}} \\
&= -r + 1 \\
&= 1 - r
\end{aligned}
$$

Again, the remaining ones follow from Part (1).

Example 2.5.2. *Use the arrangement of points in Figure 2.5.1 to verify the results of Theorem 2.5.1.*

We will look at one example for each part of the theorem. The rest are left as an exercise.First recall that we found that $(AB, CD) = 7/4$, so that is our value for r.

Interchange A and B and interchange C and D to get

$$(BA, DC) = (\overline{BD}/\overline{DA})/(\overline{BC}/\overline{CA}) = (-5/2)/(-10/7) = 7/4 = r.$$

Interchange A and B gives

$$(BA, CD) = (\overline{BC}/\overline{CA})/(\overline{BD}/\overline{DA}) = (-10/7)/(-5/2) = 4/7 = 1/r.$$

Interchange B and C to obtain

$$(AC, BD) = (\overline{AB}/\overline{BC})/(\overline{AD}/\overline{DC}) = (3/-10)/(-2/-5) = -3/4 = 1-7/4 = 1-r.$$

From Theorem 2.5.1 we can then partition the twenty-four rearrangements of (AB, CD) into six sets of four, as given in the following theorem.

Theorem 2.5.2. *If* $(AB, CD) = r$*, then*

(1) $(AB, CD) = (BA, DC) = (CD, AB) = (DC, BA) = r$

(2) $(BA, CD) = (AB, DC) = (DC, AB) = (CD, BA) = \dfrac{1}{r}$

(3) $(AC, BD) = (BD, AC) = (CA, DB) = (DB, CA) = 1 - r$

(4) $(CA, BD) = (DB, AC) = (AC, DB) = (BD, CA) = \dfrac{1}{1-r}$

(5) $(BC, AD) = (AD, BC) = (DA, CB) = (CB, DA) = 1 - \dfrac{1}{r} = \dfrac{r-1}{r}$

(6) $(CB, AD) = (DA, BC) = (AD, CB) = (BC, DA) = \dfrac{r}{r-1}$

We see that the first three are merely restatements of the results of Theorem 2.5.1. The cross ratios in (4) are obtained by applying Part (2) of Theorem 2.5.1 to (3) in Theorem 2.5.2. The results in (5) are obtained by applying Part (3) of Theorem 2.5.1 to (2) in Theorem 2.5.2. And the results in (6) are obtained by applying Part (2) of Theorem 2.5.1 to (5) of Theorem 2.5.2.

The idea of cross ratio is not restricted to a range of points. We can also define the cross ratio of a pencil of lines.

Definition 2.5.2. *The cross ratio of a pencil of four distinct coplanar lines* VA, VB, VC, VD *concurrent at an ordinary point* V *is given by*

$$V(AB, CD) = \left(\frac{\sin \overline{AVC}}{\sin \overline{CVB}}\right) \Bigg/ \left(\frac{\sin \overline{AVD}}{\sin \overline{DVB}}\right).$$

The following theorem follows immediately from the properties of parallel lines.

Theorem 2.5.3. *If four distinct parallel lines* a, b, c, d *are cut by two transversals in the points* A, B, C, D *and* A', B', C', D' *respectively, then* $(AB, CD) = (A'B', C'D')$.

Definition 2.5.3. *The cross ratio of a pencil of four distinct parallel lines,* a, b, c, d *is defined to be the cross ratio of the range of points* A, B, C, D *determined by cutting the parallel lines with any transversal.*

Theorem 2.5.4. *The cross ratio of any pencil of four distinct lines is equal to the cross ratio of the corresponding four points in which any ordinary transversal cuts the pencil.*

Proof. If the vertex V is an ideal point, then the result follows from the fact that the four lines would be parallel.

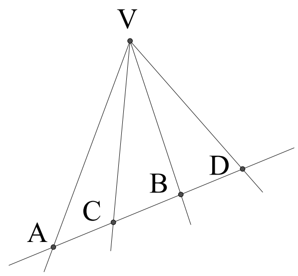


Figure 2.5.2: Cross Ratio of Four Lines

Suppose then that V is an ordinary point, and let A, B, C, D be the points in which the pencil is cut by an ordinary transversal, as in the figure above. Then we have

$$\begin{aligned} \frac{\overline{AC}}{\overline{CB}} &= \frac{VA \sin \overline{AVC}}{VB \sin \overline{CVB}} \\ \frac{\overline{AD}}{\overline{DB}} &= \frac{VA \sin \overline{AVD}}{VB \sin \overline{DVB}} \end{aligned}$$

Thus,

$$\left(\frac{\overline{AC}}{\overline{CB}}\right) \Big/ \left(\frac{\overline{AD}}{\overline{DB}}\right) = \left(\frac{\sin \overline{AVC}}{\sin \overline{CVB}}\right) \Big/ \left(\frac{\sin \overline{AVD}}{\sin \overline{DVB}}\right)$$

and hence it follows that

$$(AB, CD) = V(AB, CD).$$

The following are useful corollaries to Theorem 2.5.4.

Corollary 2.5.1. *If A, B, C, D and A', B', C', D' are two coplanar ranges on distinct bases such that $(AB, CD) = (A'B', C'D')$ and if AA', BB', CC' are concurrent, then DD' passes through the point of concurrence.*

Proof. (See the figure below) Let AA', BB', CC' meet at V. Draw VD and suppose it cuts $A'B'$ at X. Now by Theorem 2.5.4, $(AB, CD) = (A'B', C'X)$, but we know $(AB, CD) = (A'B', C'D')$. Hence $(A'B', C'X) = (A'B', C'D')$ and thus

$$\frac{\overline{A'C'}}{\overline{C'B'}} \frac{\overline{D'B'}}{\overline{A'D'}} = \frac{\overline{A'C'}}{\overline{C'B'}} \frac{\overline{XB'}}{\overline{A'X}}$$

which implies

$$\frac{\overline{D'B'}}{\overline{A'D'}} = \frac{\overline{XB'}}{\overline{A'X}}.$$

Therefore $D' = X$ and DD' passes through V.

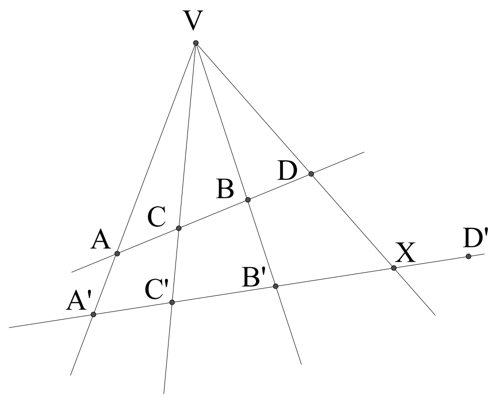


Figure 2.5.3: Corollary 1

Corollary 2.5.2. *If A, B, C, D and A', B', C', D' are two coplanar ranges on distinct bases such that $(AB, CD) = (A'B', C'D')$ and if A and A' coincide, then BB', CC', DD' are concurrent.*

Proof. (See Figure 2.5.4) Let BB', CC' meet in V. Now since A and A' coincide, AA' also meets in V. Therefore, by Corollary 2.5.1, DD' must meet AA', BB', CC' in V.

Corollary 2.5.3. *If VA, VB, VC, VD and $V'A, V'B, V'C, V'D$ are two coplanar pencils on distinct vertices such that $V(AB, CD) = V'(AB, CD)$ and if A, B, C are collinear, then D lies on the line of collinearity.*

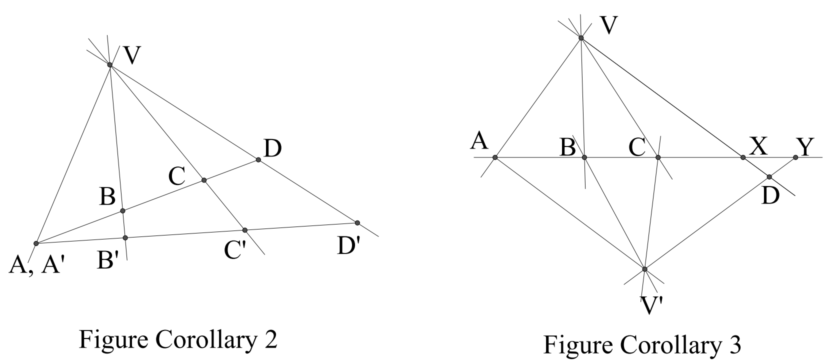


Figure 2.5.4: Corollaries 2.5.2 and 2.5.3

Proof. (See Figure 2.5.4) Suppose VD crosses AC at X and $V'D$ crosses AC at Y, as in the figure below. Now $V(AB, CD) = V'(AB, CD)$. However, $V(AB, CD) = (AB, CX)$ by Theorem 2.5.4. Furthermore, $V'(AB, CD) =$

(AB, CY). Hence $(AB, CX) = (AB, CY)$ and therefore $X \equiv Y$ which means that D is collinear with A, B and C.

Corollary 2.5.4. *If VA, VB, VC, VD and $V'A, V'B, V'C, V'D$ are two coplanar pencils on distinct vertices such that $V(AB, CD) = V'(AB, CD)$ and if A lies on VV', then B, C, D are collinear.*

Proof. In the figure below, the position of A on VV' is not important as long as A does not coincide with V or V'. Now let BC meet VV' at A'. Then $V(A'B, CD) = V'(A'B, CD)$ and A', B, C are collinear. Thus, by Corollary 2.5.3, D lies on the line of collinearity.

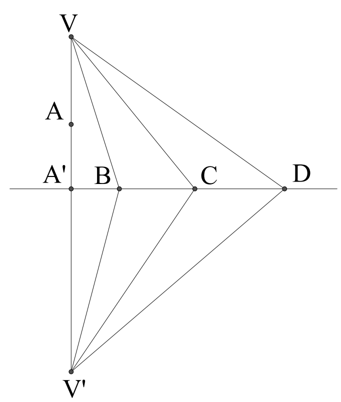


Figure 2.5.5: Corollary 2.5.4

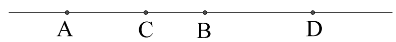
Theorem 2.5.5. *If A, B, C, D are any four distinct points on a circle, and if V and V' are any two points on the circle, then $V(AB, CD) = V'(AB, CD)$. If V should coincide with one of the points, say A, then VA is taken as the tangent to the circle at A,*

The proof of this theorem is left as an exercise. The proof follows from elementary angle relations that were discussed in Chapter 1.

Some Observations

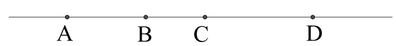
If A, B, C, D are four distinct collinear points, we can show that the pairs A, B and C, D do or do not separate each other according as (AB, CD) is negative or positive. Note that a similar statement holds for lines as well. Theorem 2.5.4 allows us to prove one and the other follows immediately. We saw earlier that

there are twenty-four arrangements of the four points A, B, C, D. All twenty-four possibilities can be reduced to three cases. For separation, there is one case that covers eight of the twenty-four possibilities.
Case 1. Separation would look like this

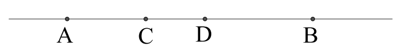


Note that we could have put any one of the 4 points in the first position, once that is done, the second position can be one of the 2 points from the second pair. There is just 1 choice for the third position, it has to be the one that pairs with the one chosen in the first position. There is only 1 choice for the fourth. Thus we take care of $4 \cdot 2 \cdot 1 \cdot 1 = 8$ of the twenty-four arrangements. Since $(AB, CD) = \frac{\overline{AC}}{\overline{CB}} \cdot \frac{\overline{DB}}{\overline{AD}}$, we see that according to the arrangement of the line, we have $\overline{AC}, \overline{CB}$ and $\overline{AD}$ are all positive and $\overline{DB}$ is negative and hence $(AB, CD) < 0$.

For non separation, all possibilities can be reduced to the following two cases: Case 2 represents eight of the twenty-four arrangements and Case 3 represents the final eight arrangements. In Case 2, $\overline{AC}, \overline{AD}$ are positive and $\overline{CB}, \overline{DB}$ are both negative and the result is $(AB, CD) > 0$. In Case 3, $\overline{AC}, \overline{CB}, \overline{AD}, \overline{DB}$ are all positive and $(AB, CD) > 0$.
Case 2.



Case 3.



Exercises

2.5.1. *Use Euler's Theorem:* $\overline{AD} \cdot \overline{BC} + \overline{BD} \cdot \overline{CA} + \overline{CD} \cdot \overline{AB} = 0$ *to show that* $(AC, BD) = 1 - r$, *where* $r = (AB, CD)$.

2.5.2. *Prove Theorem 2.5.5.*

2.5.3. *If* A, B, C, D, E *are collinear points, show that*

(a) $(AB, CE)(AB, ED) = (AB, CD)$.

(b) $(AE, CD)(EB, CD) = (AB, CD)$.

2.5.4. *If* $(AB, CD) = m$ *and* $(AB, CE) = n$, *show that* $(AC, DE) = \dfrac{n-1}{m-1}$

2.5.5. *If O, A, B, C, A', B', C' are collinear points and if*

$$\overline{OA} \cdot \overline{OA'} = \overline{OB} \cdot \overline{OB'} = \overline{OC} \cdot \overline{OC'},$$

show that $(AB', BC) = (A'B, B'C')$.

2.5.6. *Let a, b, c, d be four distinct fixed tangents to a given circle and let p be a variable fifth tangent. If p cuts a, b, c, d in A, B, C, D, show that the cross ratio (AB, CD) is a constant independent of the position of p.*

2.5.7. *Use Theorem 2.5.4 to prove Desargues Theorem.*

Historical Note.

Girard Desargues (1591-1661) was a French engineer and mathematician who is credited with inventing "projective geometry." His original work was not well know outside a small circle of French mathematicians consisting of Marin Mersenne, Rene Descartes, Étienne Pascal and Blaise Pascal, mainly because the geometric topic of the time was the newly proposed "analytic geometry." Also to its detriment was Desargues' style of writing which was not only rigorous and original but also "dense," as some have described it. It wasn't until pupils of Gaspard Monge (1746-1818) reinvented projective geometry out of "descriptive geometry" that Desargues' contribution was fully recognized.

2.6 Harmonic Division

A special case of the cross ratio arises when the pairs A, B and C, D separate each other and when the value of the cross ratio is -1. That is, when $(AB, CD) = -1$, and C and D divide AB internally and externally in the same numerical ratio.

Definition 2.6.1. *If A, B, C, D are four collinear points such that $(AB, CD) = -1$, the segment AB is said to be* **divided harmonically** *by C and D. The points C and D are called* **harmonic conjugates** *of each other with respect to A and B, and the four points A, B, C, D are said to be a* **harmonic range**. *Similarly, if $V(AB, CD) = -1$ we say that VA, VB, VC, VD constitute a* **harmonic pencil**.

The following are some elementary theorems for harmonic division.

Theorem 2.6.1. *If C and D divide AB harmonically, then A and B divide CD harmonically.*

Proof. By Theorem 2.5.1, if $(AB, CD) = -1$, then $(CD, AB) = -1$

Theorem 2.6.2. *The harmonic conjugate with respect to A and B of the midpoint of AB is the ideal point (point at infinity) on AB.*

Theorem 2.6.3. $(AB, CD) = -1$ *if and only if* $\dfrac{2}{\overline{AB}} = \dfrac{1}{\overline{AC}} + \dfrac{1}{\overline{AD}}$.

Proof. Let A, B, C, D be four collinear points with $(AB, CD) = -1$. Then

$$\left(\frac{\overline{AC}}{\overline{CB}}\right) \Big/ \left(\frac{\overline{AD}}{\overline{DB}}\right) = -1 \quad \text{or} \quad \frac{\overline{AC}}{\overline{CB}} = -\frac{\overline{AD}}{\overline{DB}}.$$

Since we need denominators of $\overline{AB}, \overline{AC}, \overline{AD}$, we invert and divide by $\overline{AB}$ to get

$$\frac{\overline{CB}}{\overline{AB} \cdot \overline{AC}} = -\frac{\overline{DB}}{\overline{AB} \cdot \overline{AD}}.$$

Using A as an origin for the numerators and writing $-\overline{DB}$ as $\overline{BD}$, we have

$$\begin{aligned} \frac{\overline{AB} - \overline{AC}}{\overline{AB} \cdot \overline{AC}} &= \frac{\overline{AD} - \overline{AB}}{\overline{AB} \cdot \overline{AD}} \\ \frac{1}{\overline{AC}} - \frac{1}{\overline{AB}} &= \frac{1}{\overline{AB}} - \frac{1}{\overline{AD}}. \end{aligned}$$

Thus,

$$\frac{2}{\overline{AB}} = \frac{1}{\overline{AC}} + \frac{1}{\overline{AD}}.$$

All the above steps are reversible, so the converse follows.

Theorem 2.6.4. $(AB, CD) = -1$ *if and only if* $\overline{OB}^2 = \overline{OC} \cdot \overline{OD}$*, where* O *is the midpoint of* AB*.*

Proof. If $(AB, CD) = -1$ then $\dfrac{\overline{AC}}{\overline{CB}} = -\dfrac{\overline{AD}}{\overline{DB}}$. If we use O as an origin and expand, we get

$$\frac{\overline{OC} - \overline{OA}}{\overline{OB} - \overline{OC}} = -\frac{\overline{OD} - \overline{OA}}{\overline{OB} - \overline{OD}},$$

but $\overline{OA} = -\overline{OB}$, so

$$\frac{\overline{OC} + \overline{OB}}{\overline{OB} - \overline{OC}} = -\frac{\overline{OD} + \overline{OB}}{\overline{OB} - \overline{OD}} = \frac{\overline{OD} + \overline{OB}}{\overline{OD} - \overline{OB}}$$

Multiplying both sides by $(\overline{OB} - \overline{OC})(\overline{OD} - \overline{OB})$ gives

$$(\overline{OC} + \overline{OB})(\overline{OD} - \overline{OB}) = (\overline{OD} + \overline{OB})(\overline{OB} - \overline{OC})$$

which simplifies to, upon multiplication,

$$\overline{OB}^2 = \overline{OC} \cdot \overline{OD}.$$

Since all of the above steps are reversible, the converse follows.

Theorem 2.6.2 gives a convenient method for relating *harmonic division* and *harmonic progression*, which we now define.

Definition 2.6.2. *The sequence of numbers* $\{a_1, a_2, a_3, \ldots, a_n\}$*is a* **harmonic progression** *if the sequence of numbers* $\left\{\dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3}, \ldots, \dfrac{1}{a_n}\right\}$*, is an arithmetic progression.*

Theorem 2.6.5. *The sequence of numbers* $\{a_1, a_2, a_3\}$ *is a harmonic progression if and only if* $\dfrac{2}{a_2} = \dfrac{1}{a_1} + \dfrac{1}{a_3}$*.*

Proof. Suppose $\{a_1, a_2, a_3\}$ is a harmonic progression. Then by the above definition, $\left\{\dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3}\right\}$ is an arithmetic progression and for some d,

$$\frac{1}{a_2} = \frac{1}{a_1} + d, \ \frac{1}{a_3} = \frac{1}{a_2} + d = \frac{1}{a_1} + d + d = \frac{1}{a_1} + 2d.$$

Thus, $\dfrac{1}{a_1} + \dfrac{1}{a_3} = \dfrac{1}{a_1} + \dfrac{1}{a_1} + 2d = 2\left(\dfrac{1}{a_1} + d\right) = \dfrac{2}{a_2}$.

On the other hand, if $\dfrac{2}{a_2} = \dfrac{1}{a_1} + \dfrac{1}{a_3}$, then $\dfrac{1}{a_2} - \dfrac{1}{a_3} = \dfrac{1}{a_1} - \dfrac{1}{a_2}$ and the difference of consecutive terms is constant and the sequence $\left\{\dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3}\right\}$ is an arithmetic progression and $\{a_1, a_2, a_3\}$ is a harmonic progression.

Theorem 2.6.6. *If* $(AB, CD) = -1$, *then* $\{\overline{AC}, \overline{AB}, \overline{AD}\}$ *is a harmonic progression.*

Let us digress a little. In calculus, when we discussed infinite series, we investigated the series $\sum_{k=1}^{\infty} \frac{1}{k}$ and even though it was a divergent series it had its own name. We called it the harmonic series. Is it related to our current discussion? If we take any three consecutive terms of the harmonic series, say $\left\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}\right\}$, their reciprocals form the arithmetic progression $\{n, n+1, n+2\}$ and thus, by definition, $\left\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}\right\}$ is a harmonic progression. We can also verify that

$$\frac{2}{\frac{1}{n+1}} = \frac{1}{\frac{1}{n}} + \frac{1}{\frac{1}{n+2}}.$$

We are used to using the terms quadrangle and quadrilateral interchangeably; however, a *complete quadrangle* and a *complete quadrilateral* are quite different in their construction. There are also some very interesting harmonic properties in both of them.

Definition 2.6.3. *A* **complete quadrangle** *(See Figure below) is the figure formed by four coplanar points, no three of which are collinear. The four points are called the* **vertices** *of the complete quadrangle, and the six lines determined by pairs of vertices are called the* **sides** *of the complete quadrangle. Pairs of sides not passing through any common vertex are called* **opposite sides** *of the complete quadrangle. The points of intersection of the three pairs of opposite sides are called the* **diagonal points** *of the complete quadrangle, and the triangle determined by the three diagonal points is called the* **diagonal 3-point** *of the complete quadrangle.*

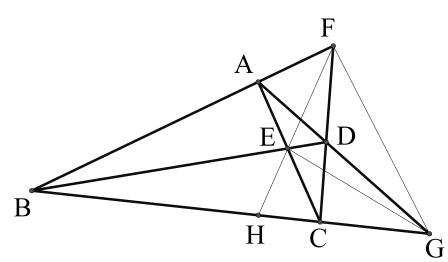


Figure 2.6.1: Complete Quadrangle

In Figure 2.6.4, the four points A, B, C, D are the vertices. The sides are the six lines AB, BC, CD, DA, AC, BD. Furthermore AB, CD are opposite sides in they don't share a vertex. AD, BC have no vertex in common and are opposite sides as are AC, BD. These opposite sides intersect in the three diagonal points

F, G, E. (**Note**. The line EH is not part of the quadrangle, but a line that will be used later.)

Definition 2.6.4. *A* **complete quadrilateral** *(See Figure below) is the figure formed by four coplanar lines, no three of which are concurrent. The four lines are called the* **sides** *of the complete quadrilateral, and the six points determined by pairs of sides are called the* **vertices** *of the complete quadrilateral. Pairs of vertices not lying on any common side are called* **opposite vertices** *of the complete quadrilateral. The lines through the three pairs of opposite vertices are called the* **diagonal lines** *of the complete quadrilateral, and the triangle determined by the three diagonal lines is called the* **diagonal 3-line** *of the complete quadrilateral.*

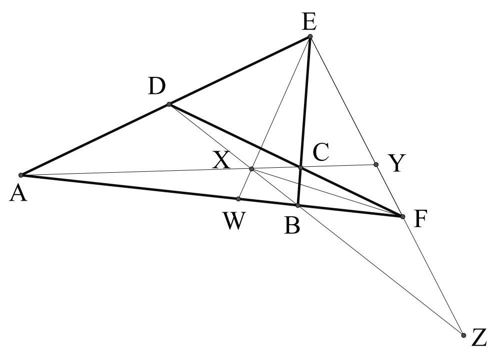


Figure 2.6.2: Complete Quadrilateral

Note. The line EW is not part of the complete quadrilateral in the figure above. It is added for later use. The reader should carefully identify each part of the complete quadrilateral given in the above figure.

We now give two theorems which identify harmonic ranges and harmonic pencils on the complete quadrilateral and complete quadrangle.

Theorem 2.6.7. *On each diagonal line of a complete quadrilateral there is a harmonic range consisting of the two vertices of the complete quadrilateral and the two vertices of the diagonal 3-line lying on it.*

Proof. In Figure 2.6.3 we need to show that $(EF, YZ) = -1$. Applying Theorem 2.5.4 and then Theorem 2.5.2 from the Cross Ratio section, we have

$$(EF, YZ) = X(EF, YZ) = X(WF, AB) = (WF, AB) = (AB, WF).$$

If we look at the figure for the quadrilateral with some of the lines removed (see the figure below), we see the figure we had for Theorem 2.3.5, the theorem about dividing a segment internally and externally in the same ratio.

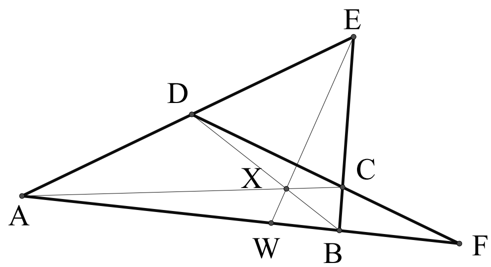


Figure 2.6.3: Harmonic Range on Diagonal lines

Thus it follows that $(AB, WF) = -1$ and hence $(EF, YZ) = -1$. Similarly, we can also show that $(DB, XZ) = (AC, XY) = -1$ in the quadrilateral.

It seems that, because of the close relationship between the quadrilateral and the quadrangle, we should have a similar result for the quadrangle involving a pencil of lines. In fact, we do have a dual statement for Theorem 2.6.7.

Theorem 2.6.8. *At each diagonal point of a complete quadrangle there is a harmonic pencil consisting of the two sides of the complete quadrangle and the two sides of the diagonal 3-point passing through it.*

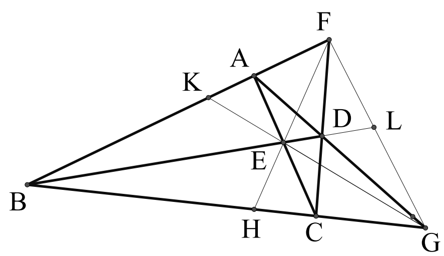


Figure 2.6.4: Harmonic Pencil on Diagonal Points

Adding the points H, K and L for reference in Figure 2.6.4 we have the harmonic pencils: $E(BC, HG), F(BD, EL), G(FK, AB)$, as well as all the other possibilities.

Exercises

2.6.1. *Justify the following methods of constructing the harmonic conjugate D of a given point C with respect to given points A and B:*

(a) *Take a point P, not on line AB and connect P to A, B, C. Through B draw the parallel to AP cutting the line PC in M and on this line mark off $BN = MB$. Then PN cuts line AB in the sought point D.*

(b) *Draw a circle on AB as diameter. If C lies between A and B, draw the perpendicular from C to AB to cut the circle in T. Then the tangent to the circle at T cuts the line AB in the sought point D. If C is not between AB, draw one of the tangents from C to the circle with T being the point of contact of the tangent. The sought point D is the foot of the perpendicular dropped from T to AB.*

(c) *Connect any point P not on line AB with A, B, C. Through A draw any line (other than AB or AP) to cut PC and PB in M and N respectively. Draw BM to cut PA in G. Now draw GN to cut AB in the sought point D.*

2.6.2. *Given a line segment AB and its midpoint M. Let P be any point not collinear with A and B. Use part (c) of the above exercise to construct a line through P that is parallel to AB.*

2.6.3. *If $(AB, CD) = -1$ and O and O' are the midpoints of AB and CD respectively, show that $(OB)^2 + (O'C)^2 = (OO')^2$.*

2.6.4. *Establish the following.*

(a) *Show that the lines joining any point on a circle to the vertices of an inscribed square form a harmonic pencil.*

(b) *Show more generally, that the lines joining any point on a circle to the extremities of a given diameter and to the extremities of a given chord perpendicular to the diameter form a harmonic pencil.*

(c) *Triangle ABC is inscribed in a circle of which DE is the diameter perpendicular to side AC. If lines DB and EB intersect AC in L and M, show that $(AC, LM) = -1$.*

(d) *Show that the diameter of a circle perpendicular to one of the sides of an inscribed triangle is divided harmonically by the other two sides.*

2.6.5. *In triangle ABC we have $(BC, PP') = (CA, QQ') = (AB, RR') = -1$. Show that AP', BQ', CR' are concurrent if and only if P, Q, R are collinear.*

2.6.6. *If P, P' divide one diameter of a circle harmonically and Q, Q' divide another harmonically, prove that P, Q, P', Q' are concyclic (lie on a circle).*

2.6.7. *Let BC be a diameter of a given circle, let A be a point on BC produced, and let P and Q be the points of contact of the tangents to the circle from point A. Show that $P(AQ, CB) = -1$.*

2.6.8. *Two circles intersect in points A and B. A common tangent touches the circles at P and Q and cuts a third circle through A and B in L and M. Prove that $(PQ, LM) = -1$.*

2.6.9. *Prove that the line through the points of contact of the incircle with two sides of a triangle cuts the third side in a point which, with the point of contact and the other two vertices on this side, forms a harmonic range.*

2.6.10. *Let AD, BE, CF be three cevian concurrent lines for triangle ABC. Let D', E', F' be the harmonic conjugates of D, E, F with respect to BC, CA, AB respectively. Show that D', E', F' are collinear.*

2.6.11. *Let A, B, C be collinear points with C between A and B. Draw the circle on AB as diameter. Let M be the midpoint of arc AB. Draw CM meeting the circle at P. Show that*

(a) *PM bisects angle APB.*

(b) *the perpendicular to PM at P meets AB in D, the harmonic conjugate of C with respect to A and B.*

2.6.12. *If $P(AB, CD) = -1$ and if PC is perpendicular to PD, show that PC and PD are bisectors of angle APB.*

2.6.13. *Given a non-square rectangle. With straightedge alone, construct an isosceles triangle with area equal to one-half that of the given rectangle.*

2.6.14. *Given a segment AB and its midpoint M. With a straightedge alone, divide the segment AB into four equal parts.*

2.6.15. *Given two parallel lines. With a straightedge alone, divide one of the lines into eight equal parts.*

2.7 Orthogonal Circles

We now turn our attention to the topic of orthogonal circles. Orthogonal circles arise in a variety of topics in geometry and here we will discuss the basic concepts we may need. Since we will be working in the extended plane, it is possible to have circles that differ from an ordinary circle (that is, a circle whose center is an ordinary point and whose radius is finite). We could possibly have a circle whose center is an ideal point. If this circle has any part in the ordinary plane, it would appear as a line. To allow for this possibility we will introduce the term "circle" (with quotation marks). Thus a "circle" will either be an ordinary circle or a straight line (a "circle" whose center is an ideal point). (Some geometers use the term *stircle* instead of "circle".)

Definition 2.7.1. *The* angles of intersection *of two coplanar curves are the angles between the two tangent lines at the points of intersection. If the angles of intersection are right angles, the curves are said to be* orthogonal.

We can summarize several basic facts about circles in the following theorem.

Theorem 2.7.1. (1) *The angles of intersection at one of the common points of two intersecting circles are equal to those at the other common point.*
(2) *If two circles are orthogonal, a radius of either, drawn to a point of intersection, is tangent to the other; conversely, if the radius of one of two intersecting circles, drawn to a point of intersection, is tangent to the other, the circles are orthogonal.*
(3) *Two circles are orthogonal if and only if the square of the distance between their centers is equal to the sum of the squares of their radii.*
(4) *If two circles are orthogonal, the center of each is outside the other.*

Proof. For Part (1), consider the following figure where CA and CB are the tangents to the circle centered at O' and DA and DB are the tangents to the circle centered at O.

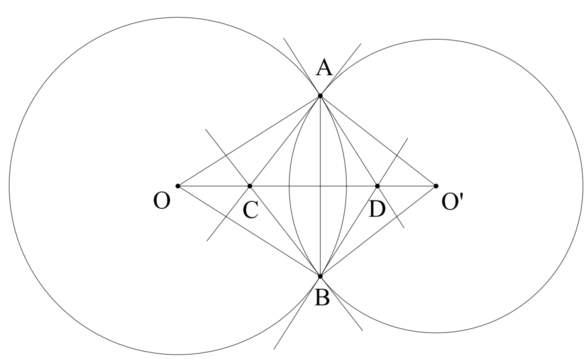


Figure 2.7.1

Triangle ACB is isosceles since CA and CB are tangents to circle O'. Therefore $\angle CAB = \angle CBA$. Likewise, triangle ADB is isosceles since DA and DB are tangents to circle O. Therefore $\angle DAB = \angle DBA$. The angle of intersection at A is $\angle CAB + \angle DAB$ which is equal to $\angle CBA + \angle DBA$, the angle of intersection at B.

Part (2) is evident since the radius of a circle drawn to a point of tangency is perpendicular to the tangent and there is only one tangent at a given point, the tangent and the radius of the other circle must be the same since the circles are orthogonal; and conversely.

For Part (3), if circles O and O' are orthogonal then by Part (2) we have that triangle OTO' is a right triangle and by the Pythagorean theorem $(OT)^2 + (O'T)^2 = (OO')^2$.

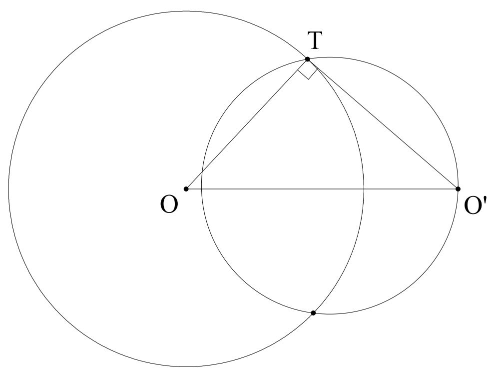


Figure 2.7.2

Part (4) is immediate, since a tangent line to a circle cannot be drawn from a point inside the circle, if two circles are orthogonal the center of each lies outside the other.

The following theorem relates orthogonal circles with harmonic ranges.

Theorem 2.7.2. *If two circles are orthogonal, then any diameter of one which intersects the other is cut harmonically by the other; conversely, if a diameter of one circle is cut harmonically by a second circle, then the two circles are orthogonal.*

Proof. Let the circles Σ and Σ' be two orthogonal circles. Let the diameter AOB of circle Σ cut the other circle, Σ', in the points C and D. Let T be one of the points of intersection of the two circles. By Theorem 2.7.1(2), OT is tangent to circle Σ' and hence $(OT)^2 = \overline{OC} \cdot \overline{OD}$. However, since OT and OB are radii of circle Σ and hence equal, we have $(OB)^2 = \overline{OC} \cdot \overline{OD}$. It now follows, from Theorem 2.6.4, that $(AB, CD) = -1$.

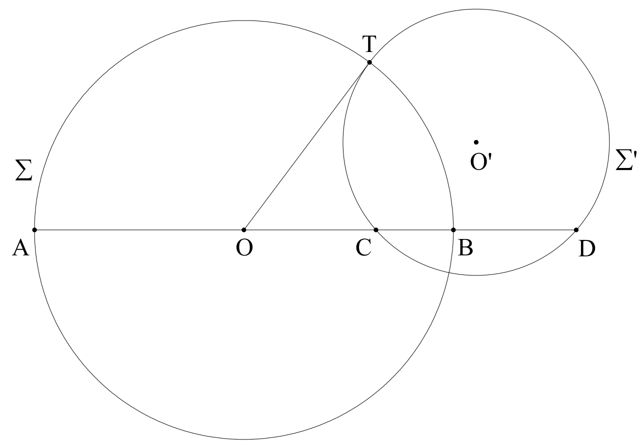


Figure 2.7.3

On the other hand, if $(AB, CD_{)} = -1$, then by Theorem 2.6.4, $(OT)^2 = (OB)^2 = \overline{OC} \cdot \overline{OD}$ and hence OT must be a tangent to circle Σ'. Therefore, by Theorem 2.7.1(2), the circles are orthogonal.

We will now look at some results that use the concept of a "circle." In order for some of our statements to make sense, we must agree that two straight lines (which might represent "circles") are tangent if they either coincide or they are parallel. Furthermore, an ordinary circle and a straight line ("circle") are orthogonal if and only if the line passes through the center of the circle; that is, coincides with a diameter.

Theorem 2.7.3. *There is one and only one "circle" orthogonal to a given circle Σ and passing through two given interior points A and B of Σ.*

Let Σ be a circle with center O and let A and B be two points interior to Σ. If A, O, B are collinear, then the "circle" is the straight line through A and B. If A, O, B are not collinear, then let A' be the harmonic conjugate of A with respect to the diameter of Σ passing through A. (See Figure 2.7.4.) Now A, B, A' are not collinear and hence determine a unique circle Σ' and by Theorem 2.7.2 it is orthogonal to Σ. Hence we have at least one "circle" through the points A and B orthogonal to Σ.

To show that there is only one such "circle," let Σ' represent any "circle" through A and B orthogonal to Σ. If Σ' is a straight line, then it must be a

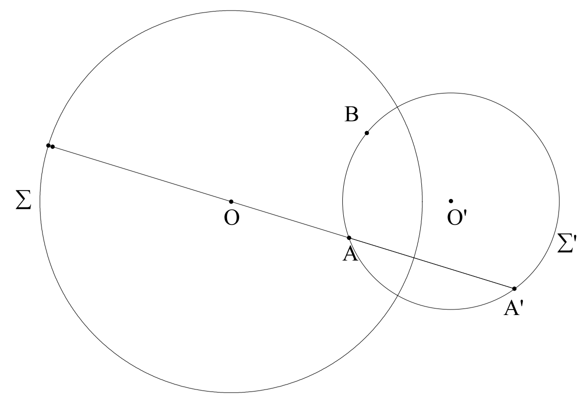


Figure 2.7.4

diameter of Σ and coincide with the diameter through A, O, B we found earlier. If Σ' is a circle, then by Theorem 2.7.2 it must also pass through A' and it is the circle we found earlier passing through A, B, A'. Therefore the circle is unique, since three noncollinear points uniquely determine a circle.

Theorem 2.7.4. *There is a unique "circle" orthogonal to a given circle, Σ, and tangent to a given line ℓ at an ordinary point A of ℓ not on Σ.*

Proof. If O lies on ℓ, then ℓ is the unique "circle" satisfying the stated conditions. If O does not lie on ℓ, then draw the diameter through A (see Figure 2.7.5 below). Find B, the harmonic conjugate of A with respect to C and D, the endpoints of the diameter. Draw the perpendicular to ℓ at A to determine a radius line from the point of tangency A. Construct the perpendicular bisector of AB at M meeting the perpendicular drawn from A at O'. Recall the perpendicular bisector of a chord of a circle passes through the center of the circle. This locates the center of the circle with $O'A$ as the radius. This circle is unique.

The concept of orthogonal circles was introduced in the nineteenth century along with other concepts we will be discussing in this chapter. In particular, was the concept of the power of a point with respect to a circle. The basic idea was conveyed in the following theorem.

Theorem 2.7.5. *If P is a fixed point in the plane of a given circle Σ, and if a variable line ℓ through P intersects Σ in points A and B, then the product $\overline{PA} \cdot \overline{PB}$ is independent of the position of ℓ.*

Definition 2.7.2. *The* power of a point with respect to a circle *is the product of the directed distances of the point from any two points on the circle and collinear with it.*

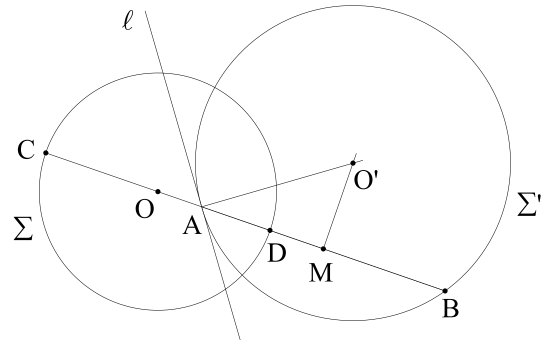


Figure 2.7.5

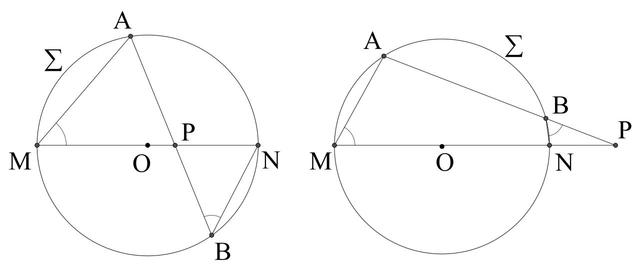


Figure 2.7.6

In Figure A, on the next page, the power of the point P with respect to the circle centered at O is $\overline{PQ} \cdot \overline{PR} > 0$. In Figure B, P and Q coincide, so that $\overline{PQ} \cdot \overline{PR} = 0$ since $\overline{PQ} = 0$. In Figure C, we see that $\overline{PQ} \cdot \overline{PR} < 0$, since $\overline{PQ}$ and $\overline{PR}$ are directed in opposite directions.

In Figure 2.7.6, we see that if the point P lies outside the circle, its power with respect to the circle is also equal to the square of the tangent from the point P to the circle. As an exercise, one might show that if P is inside the circle, then its power with respect to the circle is the negative of the square of half the chord perpendicular to the diameter passing through P. These observations lead to the following theorem.

Theorem 2.7.6. *Let P be a point in the plane of a circle Σ of center O and radius r. Then the power of P with respect to Σ is equal to $(OP)^2 - r^2$.*

Proof. For the case where P is outside the circle, we see from Figure 2.7.8, below left, that, using $(PT)^2$ as the power of the point P, $(PT)^2 + r^2 = (OP)^2$. Hence we have $(PT)^2 = (OP)^2 - r^2$.

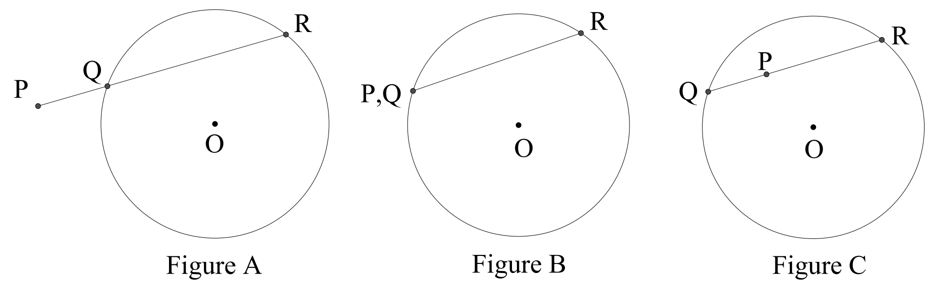


Figure 2.7.7: Powers of a Point

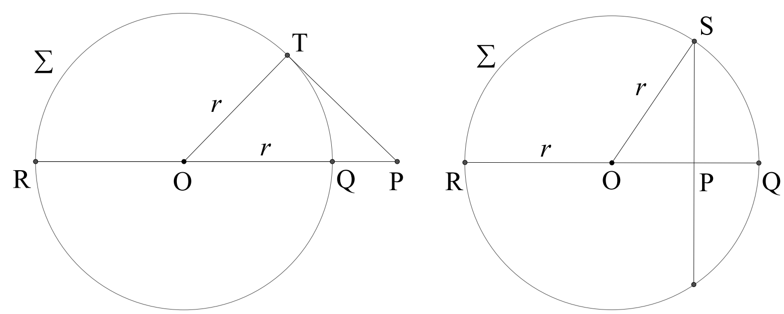


Figure 2.7.8

For the case where P is inside Σ, consider Figure 2.7.8, above right. Recall that if P is inside the circle, then its power with respect to the circle is the negative of the square of half the chord perpendicular to the diameter passing through P. Hence the power of P is given by $-(PS)^2$ and since $r^2 = (OP)^2 + (PS)^2$, it follows that $-(PS)^2 = (OP)^2 - r^2$.

(Note: We can also show this without using the above argument. Since the power is independent of the line used, we use a diameter. Then $\overline{OQ} = r = \overline{OP}+\overline{PQ}$ and $\overline{PR} = \overline{PO}+\overline{OR} = \overline{PO}-r$. Thus, $\overline{PQ} = -\overline{OP}+r$, $\overline{PR} = -\overline{OP}-r$ and $\overline{PQ} \cdot \overline{PR} = (OP)^2 - r^2$.)

Finally, using the concept of the power of a point with respect to a circle, we can get an equivalent statement for Theorem 2.7.1(3).

Theorem 2.7.7. *Two circles are orthogonal if and only if the power of the center of either with respect to the other be equal to the square of the corresponding radius.*

Proof. Let Σ and Σ' be the two circles with centers O and O' and radii r and r'. The power of O with respect to Σ' is, by Theorem 2.7.6, $(O'O)^2 - r'^2$. If $(O'O)^2 - r'^2 = r^2$ then $(O'O)^2 = r'^2 + r^2$ and by Theorem 2.7.1(3) the circles are orthogonal.

On the other hand, suppose the circles are orthogonal. Then by Theorem 2.7.1(3), $(O'O)^2 = r'^2 + r^2$ and $r^2 = (O'O)^2 - r'^2$. By Theorem 2.7.6, $(O'O)^2 - r'^2$ is the power of O with respect to Σ', and the power of O with respect to Σ' is r'^2.

Exercises

2.7.1. *Show that if d is the distance between the centers of two intersecting circles, c the length of their common chord, r and r' their radii, then the circles are orthogonal if and only if $cd = 2rr'$.*

2.7.2. *If a line drawn through a point of intersection of two circles meets the circles again in P and Q respectively, show that the circles with centers P and Q, each orthogonal to the other circle, are orthogonal to each other.*

2.7.3. *Let H be the orthocenter of a triangle ABC. Show that the circles on AH and BC as diameters are orthogonal.*

2.7.4. *If AB is a diameter and M any point of a circle of center O, show that the two circles AMO and BMO are orthogonal.*

2.7.5. *If the quadrilateral whose vertices are the centers and the points of intersection of two circles is cyclic, prove that the circles are orthogonal.*

Definition 2.7.3. *Four points are said to be an* orthocentric group of points *if each is the orthocenter of the triangle formed by the remaining three.*

2.7.6. *Show that the vertices and the orthocenter of a triangle form an orthocentric group of points.*

2.7.7. *Let two orthogonal circles be given. If a diameter AB of the first is perpendicular to the diameter CD of the second, show that the points A, B, C, D form an orthocentric group of points.*

2.8 The Radical Axis of a Pair of Circles

In this material we will revisit powers of points and orthogonal circles. One might wonder if it is possible for a point to have the same power with respect to two distinct circles. If so, can the locus for all such points be found? One might also wonder how can a circle be constructed that is orthogonal to two given circles, if possible.

Definition 2.8.1. *The locus of a point whose powers with respect to two circles are equal is called the* radical axis *of the two given circles.*

We saw that the power of a point with respect to a circle Σ is positive if the point is exterior to Σ, negative if the point is inside Σ and 0 if the point is on Σ. This would lead one to speculate that the radical axis for two intersecting circles would be the line through the points of intersection.

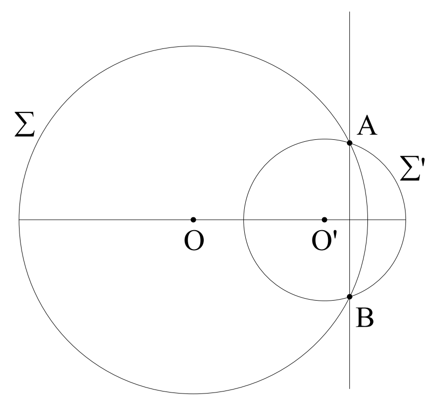


Figure 2.8.1

In the above figure, all points on the line through A and B have the same power with respect to each circle, since by definition:

> The power of a point with respect to a circle is the product of the directed distances of the point from any two points on the circle and collinear with it.

For any point on the line through A and B, A and B are the only two points on the circles and collinear with it. Now the line through A and B is perpendicular to the line of centers OO' and this suggests the following.

Theorem 2.8.1. *The radical axis of two nonconcentric circles is a straight line perpendicular to the line of centers of the two circles.*

Proof. Let O and O' be the centers and r and r' be the radii of two nonconcentric circles Σ and Σ'. Let P be a point on the radical axis of Σ and Σ' and let Q be the foot of the perpendicular from P to OO', as in the figure below.

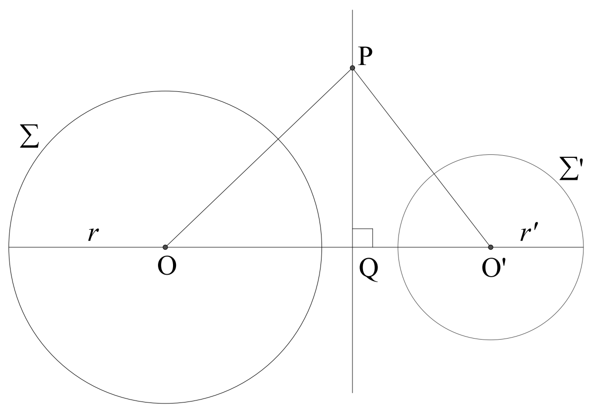


Figure 2.8.2

We know, by Theorem 2.7.6, that the power of P with respect to Σ is $(OP)^2 - r^2$ and the power of P with respect to Σ' is $(O'P)^2 - r'^2$. Because P is on the radical axis, we have that

$$(OP)^2 - r^2 = (O'P)^2 - r'^2. \tag{2.13}$$

Triangles PQO and PQO' are right triangles, so, by the Pythagorean theorem, we have that $(OP)^2 = (PQ)^2 + (OQ)^2$ and $(O'P)^2 = (PQ)^2 + (O'Q)^2$. Substituting into (2.13) we get

$$(PQ)^2 + (OQ)^2 - r^2 = (PQ)^2 + (O'Q)^2 - r'^2$$

and subtracting $(PQ)^2$ from both sides gives

$$(OQ)^2 - r^2 = (O'Q)^2 - r'^2. \tag{2.14}$$

Thus, Q is on the radical axis. Rewriting (2.14) gives

$$(OQ)^2 - (O'Q)^2 = r^2 - r'^2,$$

or

$$\begin{aligned}
(\overline{OQ} + \overline{O'Q})(\overline{OQ} - \overline{O'Q}) &= r^2 - r'^2 \\
(\overline{OQ} + \overline{O'Q})(\overline{OQ} + \overline{QO'}) &= r^2 - r'^2 \\
(\overline{OQ} + \overline{O'Q})\overline{OO'} &= r^2 - r'^2
\end{aligned}$$

And we have
$$\overline{OQ} + \overline{O'Q} = \frac{r^2 - r'^2}{\overline{OO'}}. \tag{2.15}$$
Now there is only one point Q on OO' satisfying (2.15). For if there were another such point R we would have
$$\overline{OQ} + \overline{O'Q} = \overline{OR} + \overline{O'R},$$
or
$$(\overline{OR} + \overline{RQ}) + \overline{O'Q} = \overline{OR} + (\overline{O'Q} + \overline{QR}),$$
or
$$\overline{OR} + \overline{RQ} + \overline{O'Q} = \overline{OR} + \overline{O'Q} + \overline{QR}$$
$$\overline{RQ} = \overline{QR}$$
and $\overline{RQ} = \overline{QR}$ implies $\overline{RQ} = 0$, which means that R coincides with Q. It then follows that if a point is on the radical axis of Σ and Σ', it lies on the perpendicular to the line of centers at the point Q. Conversely, by reversing the above steps, we can show that any point on the perpendicular to OO' at Q lies on the radical axis of the two circles. Therefore the radical axis of the two circles is the perpendicular to OO' at the point Q.

Theorem 2.8.1 excluded concentric circles. What should the radical axis of two concentric circles of unequal radius be? If we look at equation (2.15) in the proof of Theorem 2.8.1, we see that as O' approaches O it forces Q to approach an ideal point and thus the radical axis would be the line at infinity. If the concentric circles have the same radius, we consider their radical axis to be undefined and we rule out this case in any discussion of radical axes.

Construction. Construct the radical axis for two given, nonconcentric circles.

Solution. We have already seen the case for intersecting circles. There are two remaining and the same technique works for both.

> **Case 1.** Circles that are nonintersecting and are exterior to each other. Let Σ and Σ' be the two given circles. Draw any circle Σ'' that intersects both given circles, as in Figure 2.8.3. Let A, B be the points of intersection of Σ'' and Σ and let C, D be the points of intersection of Σ'' and Σ'. Draw the lines on A, B and C, D. Where these intersect, say P, Draw the perpendicular to OO'.

> **Case 2.** One circle lies within the other, but the circles are not concentric. The construction is the same as above, as given in Figure 2.8.4.

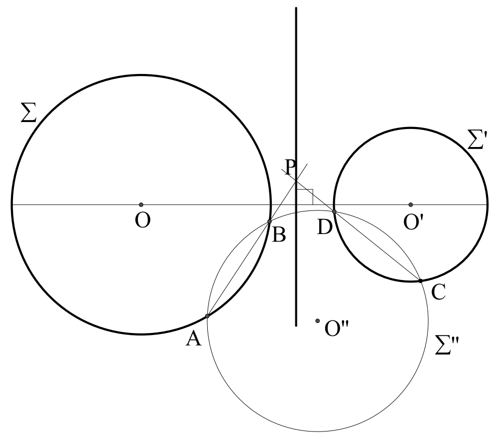


Figure 2.8.3: Non-Intersecting Circles

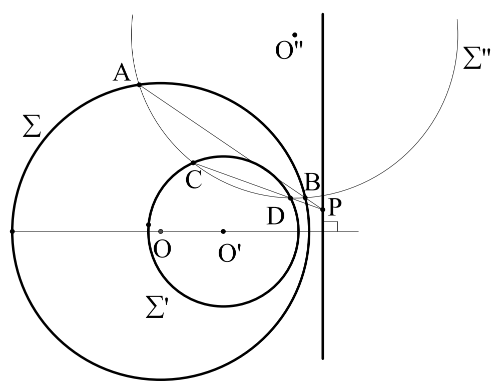


Figure 2.8.4: Circle Within Circle

Theorem 2.8.2. *The radical axes of three circles with noncollinear centers, taken in pairs, are concurrent.*

Proof. Let Σ, Σ', Σ'' be the three circles. If P is the point of intersection of the radical axis of Σ and Σ' with the radical axis of Σ' and Σ'', then the power of P with respect to Σ, Σ', Σ'' is the same. Therefore P is on the radical axis of Σ and Σ''. See Figure 2.8.5

Note. The circles of Theorem 2.8.2 can be intersecting or nonintersecting. The only requirement is that their centers are not collinear.

Definition 2.8.2. *The point of concurrence of the radical axes of three circles with noncollinear centers, taken in pairs, is called the* radical center *of the three*

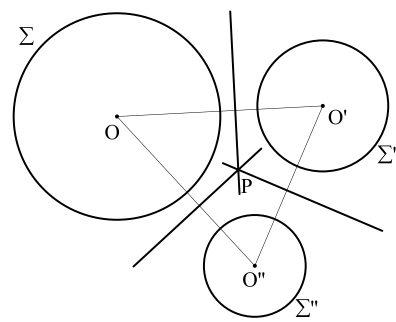


Figure 2.8.5: Radical Center

circles.

Theorem 2.8.3. (1) *The center of a circle that cuts each of two circles orthogonally lies on the radical axis of the two circles.* (2) *If a circle whose center lies on the radical axis of two circles is orthogonal to one of them, it is also orthogonal to the other.*

Proof. In Figure 2.8.6 suppose Σ'' is orthogonal to both circles, Σ and Σ'. By Theorem 2.7.7, the power of P with respect to both Σ and Σ' is r''^2. Thus, since the power of P with respect to both circles is the same, P must be on the radical axis for Σ and Σ'.

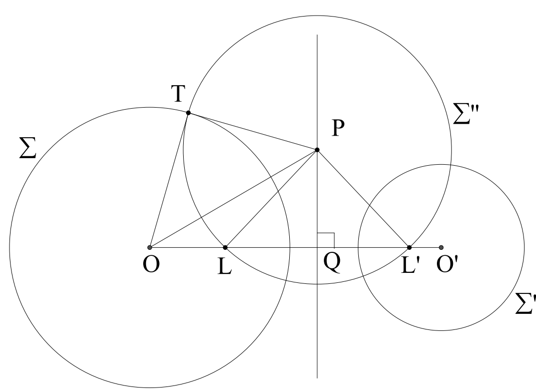


Figure 2.8.6

Next, suppose the center of Σ'' is on the radical axis for Σ and Σ' and is orthogonal to Σ. Since PT is tangent to Σ and $PT = r''$, the power of P with respect to Σ is $PT^2 = r''^2$. However, siince P is on the radical axis for Σ and Σ', its power with respect to Σ' must also be r''^2, which by Theorem 2.7.7, says Σ'' and Σ' are orthogonal.

Note. Part (1) of Theorem 2.8.3 tells us that the locus of the centers of circles that are orthogonal to two given circles is the radical axis of the two circles. Thus, if we are asked to construct a circle that is orthogonal to two given circles, the first thing we do is construct the radical axis. We can then choose any point on the radical axis and construct a circle with that point as center that is orthogonal to either of the two given circles. The Part (2) of Theorem 2.8.3 guarantees it is orthogonal to the other given circle.

Theorem 2.8.4. *All the circles that cut each of two nonintersecting circles orthogonally intersect the line of centers of the two given circles in the same two points.*

Proof. Let Σ and Σ' be the two given nonintersecting circles in Figure 2.8.7, with centers O and O'

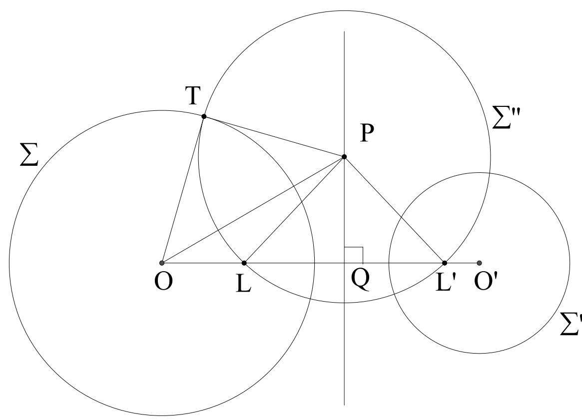


Figure 2.8.7: Non-Intersecting Circles

Let a circle with center P cut the two given circles orthogonally. Then, by Theorem 2.8.3, P lies on the radical axis of Σ and Σ'. Using Figure 2.8.7 as a reference, we have that $OQ > OT$, since the radical axis is outside both circles. Now in right triangles POQ and POT, we have that

$$(PO)^2 = (PT)^2 + (OT)^2 = (PQ)^2 + (OQ)^2$$

and, thus,

$$(PT)^2 = (PQ)^2 + (OQ)^2 - (OT)^2$$

and we know $(OQ)^2 - (OT)^2 > 0$, so $(PT)^2 > (PQ)^2$ and $PT > PQ$. Therefore, the orthogonal circle must cut OO' in two distinct points, say L and L'. Now

$$(PL)^2 = (LQ)^2 + (QP)^2 \tag{2.16}$$

and also, since PL and PT are radii of the orthogonal circle,

$$(PL)^2 = (PT)^2 = (PO)^2 - (OT)^2 = (OQ)^2 + (QP)^2 - (OT)^2. \tag{2.17}$$

Next, substituting results from (2.17) into (2.16), gives

$$(OQ)^2 + (QP)^2 - (OT)^2 = (LQ)^2 + (QP)^2$$

or

$$(LQ)^2 = (OQ)^2 - (OT)^2. \tag{2.18}$$

Equation (2.18) shows that the position of L with respect to Q is independent of the position of P. Hence every circle orthogonal to Σ and Σ' passes through L and, by symmetry, L'.

NOTE: L and L' are harmonic conjugates with respect to the diameter points of the circle Σ as well as for the circle Σ'.

Theorem 2.8.5. *A circle that cuts each of two intersecting circles orthogonally does not intersect the line of centers of the two given circles.*

Proof. Let Σ and Σ' be the two given intersecting circles, with centers O and O'.

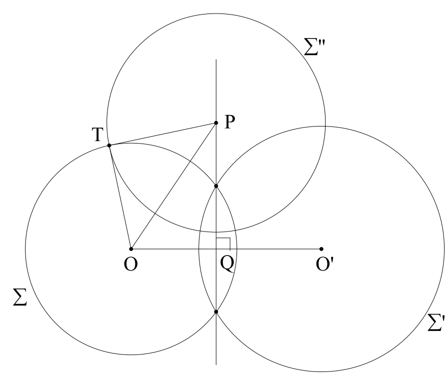


Figure 2.8.8: Intersecting Circles

Here we have $OQ < OT$ and using the argument in the proof of Theorem 2.8.4, we have

$$(OP)^2 = (PT)^2 + (OT)^2 = (PQ)^2 + (OQ)^2$$

and

$$(PQ)^2 = (PT)^2 + (OT)^2 - (OQ)^2$$

so $PQ > PT$. Since PT is the radius of the common orthogonal circle, it fails to intersect OO'

Definition 2.8.3. *A set of circles is said to form a* coaxial pencil of circles *if the same straight line is the radical axis of any two circles of the set. The straight line is called the* radical axis *of the coaxial pencil of circles.*

Theorem 2.8.6. (1) *The centers of the circles of a coaxial pencil are collinear.* (2) *If two circles of a coaxial pencil intersect, every circle of the coaxial pencil passes through the same two points of intersection; if two circles of a coaxial pencil are tangent, all circles of the coaxial pencil are tangent to one another at the same point; if two circles of a coaxial pencil do not intersect, no two circles of the coaxial pencil intersect.* (3) *The radical axis of a coaxial pencil of circles is the locus of a point whose powers with respect to all the circles of the pencil are equal.*

Proof. (1) Suppose ℓ is the radical axis for the coaxial pencil. If Σ_1 and Σ_2, with centers O_1 and O_2 are two circles in the coaxial pencil, then, by Theorem 2.8.1, O_1O_2 is perpendicular to ℓ. Now suppose Σ_3, with center O_3, is another circle of the pencil. Then O_1O_3 is perpendicular to ℓ. Since through point O_1 there can be only one perpendicular to ℓ, O_3 must be collinear with O_1 and O_2. This is true of any other circle in the coaxial pencil and, hence, the centers of all circles of the pencil are collinear.

(2) If two circles of the coaxial pencil intersect, then the common chord is the radical axis for the two circles and also for the entire coaxial pencil. Thus, for any other circle to share that radical axis, the circle must go through the common points of intersection. If two of the circles are tangent, their radical axis, and the radical axis for the pencil, must be the line through the point of tangency and perpendicular to the line of centers. Thus, any other circle in the pencil must have the same radical axis with the two tangent circles. In order for this to happen, the circle must be tangent to the two original circles at the original point of tangency. If two of the circles in the pencil do not intersect, the radical axis is a line ℓ perpendicular to the line of centers and outside both circles. If any two circle of the pencil intersect, then by the first part of (2), the radical axis would be the common chord and there would be powers of point that are negative, which cannot happen.

(3) This follows immediately from the definition of a radical axis.

From Theorem 2.8.6 we see that there are three possible types of coaxial pencils of circles: an *intersecting coaxial pencil of circles*; a *tangent coaxial pencil of circles*; and a *nonintersecting coaxial pencil of circles.*

Illustrations for Theorem 2.8.6 can be found in Appendix C.

We close with the following theorem about coaxial pencils of circles.

Theorem 2.8.7. (1)*All the circles orthogonal to two given nonintersecting circles belong to an intersecting coaxial pencil whose line of centers is the radical axis of the two given circles.* (2) *All the circles orthogonal to two given tangent circles belong to a tangent coaxial pencil whose line of centers is the common* *tangent to the two given circles.* (3) *All the circles orthogonal to two given intersecting circles belong to a nonintersecting coaxial pencil whose line of centers is the line of the common chord of the two given circles.*

Theorem 2.8.7 is also illustrated in Appendix C. The circles with centers O_1 and O_2 are the given circles.

EXERCISES

2.8.1. *Show that if the radical center of three circles with noncollinear centers is exterior to each of the circles, it is the center of a circle orthogonal to all three circles. (This circle is called the* radical circle *of the three circles.*

2.8.2. *Prove that the radical axis of two circles having a common tangent bisects the segment on the common tangent determined by the points of contact of the tangent.*

2.8.3. *Prove that the radical center of three circles constructed on the sides of a triangle as diameters is the orthocenter of the triangle.*

2.8.4. *Let AD, BE, CF be three cevian lines of triangle ABC. Prove that the radical center of the circles constructed on AD, BE, CF as diameters is the orthocenter of the triangle.*

2.8.5. *If a point is considered to be a circle of radius zero, find the radical axis of a point-circle and a circle.*

2.8.6. *Through a given point draw a circle that is orthogonal to two given circles.*

2.8.7. *Through a given point draw a circle that is coaxial with two given circles.*

2.8.8. *Prove that if each of a pair of circles cut each of a second pair orthogonally, then the radical axis of either pair is the line of centers of the other.*

2.8.9. *Construct a non-intersecting coaxial pencil of circles. Explain the process you used. See Figure C-3 in Appendix C.*

Chapter 3

Transformations of the Plane

3.1 Fundamental Transformations of the Plane

Definition 3.1.1. *A* transformation *is a one-to-one mapping of a set X onto a set Y.*

> **Note.** A mapping of X onto Y is one-to-one if distinct elements of X have distinct images in Y.

Definition 3.1.2. *If X and Y are the same set, then the mapping is a transformation of X* onto itself*. Any point that maps to itself under a transformation is called a fixed point or an* invariant point*. If every element of the transformation is invariant, then the transformation is said to be the* identity transformation *and is denoted by I.*

We now look at some special transformations of the plane.

Definition 3.1.3. *Let $\overline{AB}$ be a directed line segment (vector) in the plane. A* translation*, $T(AB)$ is a transformation of the plane onto itself which carries each point P of the plane onto a point P' of the plane such that $\overline{PP'}$ is equal and parallel to $\overline{AB}$. The directed segment $\overline{AB}$ is called the vector of the translation.*

Example 3.1.1. Figure 3.1.1*, below, shows a point P being translated to its image P'.*

Often it is easier to see the effects of a transformation by observing its effect on a region. From this point on we will use a triangle as the object of our transformations. In Figure 3.1.2 we see the results of translating a triangle PQR by the translation $T(AB)$.

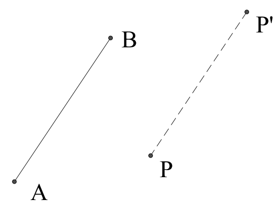


Figure 3.1.1: Translation of a point

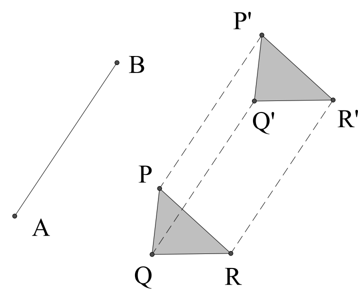


Figure 3.1.2: Translation of a triangle

Definition 3.1.4. *Let O be a fixed point in the plane and θ a given directed angle. A* rotation, *$R(O, \theta)$, is a transformation of the plane onto itself which carries each point P of the plane onto the point P' of the plane such that $OP' = OP$ and $\angle\overline{POP'} = \theta$. Point O is called the* center *of the rotation and θ is called the* angle *of the rotation.*

[NOTE: $\angle\overline{POP'}$ means the angle with vertex O and P on the initial side of the angle and P' on the terminal side of the angle.]

Example 3.1.2. Figure 3.1.3 *shows a triangle PQR rotated about the point O through an angle of $+60°$. That is, $R(O, +60°)$, where $+60°$ means an angle of $60°$ measured in the usual positive direction (counterclockwise and, from now on, if no sign appears with the angle, it is assumed to be positive). $\angle\overline{POP'} = 60°, \angle\overline{QOQ'} = 60°$ and $\angle\overline{ROR'} = 60°$. The angle $\angle\overline{POP'} = 60°$ is identified on the drawing.*

Definition 3.1.5. *Let O be a fixed point of the plane. The transformation of the plane onto itself which carries each point P of the plane to a point P' such that O is the midpoint of PP' is called a* reflection in the point O *and is denoted by $R(O)$. This transformation is also called a* half-turn about the point O. *The point O is called the center of the reflection.*

Example 3.1.3. Figure 3.1.4 *illustrates the transformation $R(O)$ which reflects the triangle ABC through O to get triangle $A'B'C'$.*

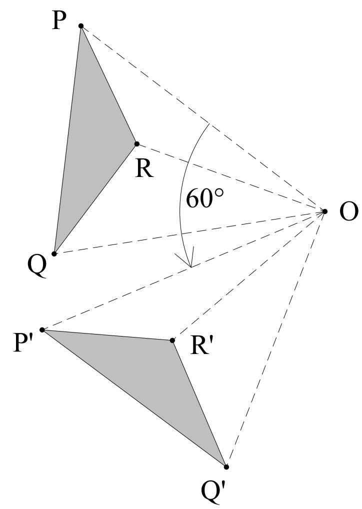


Figure 3.1.3: Rotation of a triangle

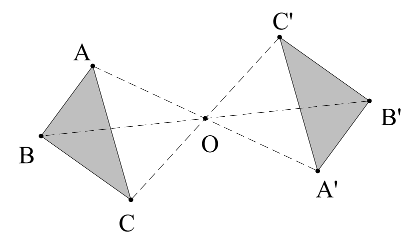


Figure 3.1.4: Reflection of a triangle through a point O

Definition 3.1.6. *Let ℓ be a fixed line of the plane. A* reflection in a line $R(\ell)$, *is the transformation of the plane onto itself which carries each point P of the plane onto the point P' of the plane such that ℓ is the perpendicular bisector of the line PP'. The line ℓ is called the* axis *(or* mirror*) of the reflection.*

Example 3.1.4. *In* Figure 3.1.5*, triangle ABC is reflected through the line ℓ to triangle $A'B'C'$. Note that ℓ is the perpendicular bisector of AA', BB' and CC'.*

The above transformations are what we call *isometries*. They all preserve distance and angle measure. That is, under these transformations, figures are mapped onto congruent figures.

Definition 3.1.7. *Let O be a fixed point of the plane and k be a given nonzero real number. The* homothety *(or* dilation*, or* expansion*, or* contraction*) $H(O,k)$ is defined to be the transformation of the plane onto itself which carries each*

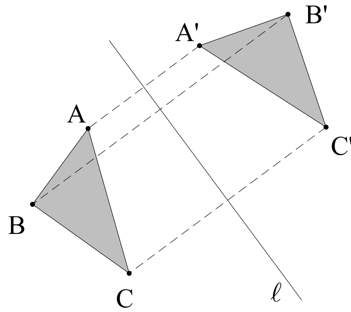


Figure 3.1.5: Reflection of a triangle through a line ℓ

point P of the plane onto a point P' (collinear with P and O) of the plane such that $\overline{OP'} = k\overline{OP}$. The point O is called the center *of the homothety and k is called the* ratio *of the homothety.*

Example 3.1.5. $H(O, \frac{1}{2})$. *In* Figure 3.1.6*, the points A, B, C are mapped onto the points A', B', C', each of which is halfway between the original point and the center of the homothety, O. That is, $\overline{OA'} = \frac{1}{2}\overline{OA}$, $\overline{OB'} = \frac{1}{2}\overline{OB}$, and $\overline{OC'} = \frac{1}{2}\overline{OC}$. In fact, under $H(O, \frac{1}{2})$, every point P of the plane is mapped onto a point P' that is halfway between O and P.*

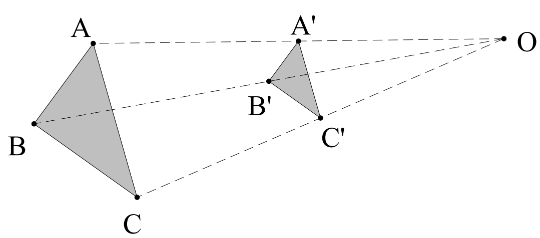


Figure 3.1.6: Transforming a triangle by $H(O, \frac{1}{2})$

Note. If $k > 0$ then the image of a point P will be on the same side of O as P and if $k < 0$, then the image will be on the opposite side of O; that is, O will be between P and P'.

Example 3.1.6. $H(O, -2)$. *In Figure 3.1.7, the points A, B, C are mapped onto the points A', B', C', each of which is on the opposite side of the center of the homothety, O, and twice as far from O as A, B, C. That is, $\overline{OA'} = -2 \cdot \overline{OA}$, $\overline{OB'} = -2 \cdot \overline{OB}$, and $\overline{OC'} = -2 \cdot \overline{OC}$.*

Example 3.1.7. $H(O, 2.25)$. *In* Figure 3.1.8*, the points A, B, C are mapped onto the points A', B', C', each of which is* 2.25 *times as far from the center of*

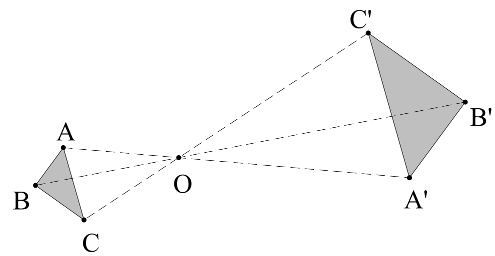


Figure 3.1.7: Transforming a triangle by $H(O, -2)$

the homothety, O, as the original points and on the same side of O. That is, $\overline{OA'} = 2.25 \cdot \overline{OA}$, $\overline{OB'} = 2.25 \cdot \overline{OB}$, and $\overline{OC'} = 2.25 \cdot \overline{OC}$.

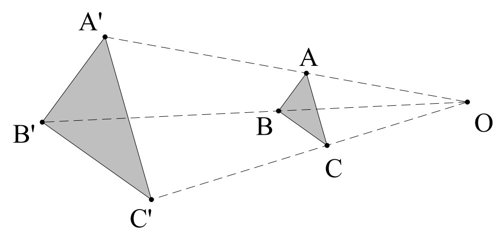


Figure 3.1.8: Transforming a triangle by $H(O, 2.25)$

Definition 3.1.8. *Let T_1 be a transformation of a set X onto a set Z and T_2 a transformation from the set Z onto a set Y. (The sets X, Y, Z may all be the same set.) The* product transformation *$T = T_2T_1$ of set X onto the set Y is the result of performing the transformation T_1 followed by the transformation T_2. If such a transformation exists, we say that T_2 is* compatible *with T_1.*

NOTE: The product transformation is analogous to composite functions we encountered in algebra and calculus. We could call them composite mappings here, but in geometry they are more often referred to as product transformations.

Theorem 3.1.1. *If a transformation T_2 is compatible with a transformation T_1, then $(T_2T_1)^{-1} = T_1^{-1}T_2^{-1}$.*

Example 3.1.8. *In* Figure 3.1.9*, triangle ABC is transformed, by the rotation $R(O, -100°)$, to triangle $A'B'C'$. Then triangle $A'B'C'$ is mapped onto triangle $A''B''C''$ by the reflection $R(\ell)$. Thus, we would say that triangle $A''B''C''$ is the image of triangle ABC under the product transformation $R(\ell)R(O, -100°)$. In the figure $\angle\overline{COC'} = -100°$ is marked.*

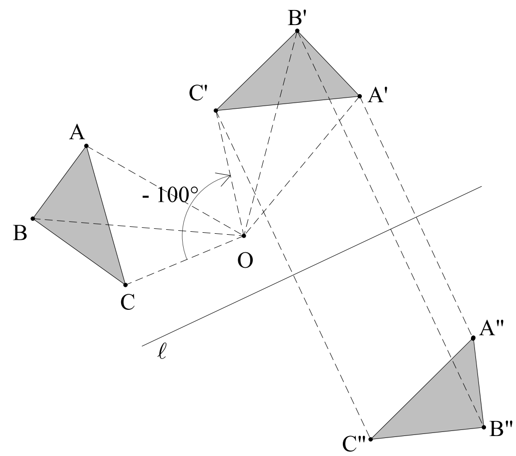


Figure 3.1.9: Transforming a triangle by $R(\ell)R(O,-100°)$

Definition 3.1.9. *Let ℓ be a fixed line of the plane and $\overline{AB}$ a given directed segment on the line ℓ. The product $R(\ell)T(AB)$ defines a transformation called the* glide-reflection*, $G(\ell, AB)$. The line ℓ is called the* axis *of the glide-reflection and the directed segment $\overline{AB}$ on ℓ is called the* vector *of the glide-reflection.*

Example 3.1.9. *In* Figure 3.1.10*, triangle PQR is mapped onto triangle $P'Q'R'$ by the translation $T(AB)$, then triangle $P'Q'R'$ is mapped onto triangle $P''Q''R''$ by the reflection $R(\ell)$. The end result is that the glide-reflection $G(\ell, AB)$ maps triangle PQR onto triangle $P''Q''R''$.*

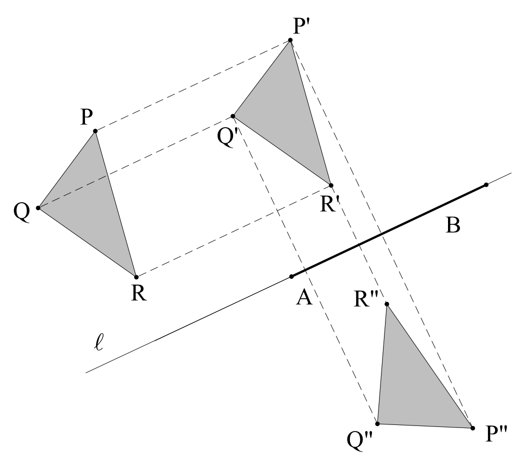


Figure 3.1.10: Transforming a triangle by $G(\ell, AB)$

Definition 3.1.10. *Let ℓ be a fixed line of the plane, let O be a fixed point on line ℓ and let k be a given nonzero number. The product $R(\ell)H(O,k)$ defines a transformation called the* dilation-reflection*, $S(O,k,\ell)$. The line ℓ is called the* axis*, O is called the* center *and k is called the* ratio *of the dilation-reflection.*

Example 3.1.10. *In* Figure 3.1.11*, triangle ABC is mapped onto triangle $A'B'C'$ by the dilation $H(O,\frac{1}{2})$ and then triangle $A'B'C'$ is mapped onto triangle $A''B''C''$ by the reflection $R(\ell)$. The end result is that triangle ABC is mapped onto triangle $A''B''C''$ by the dilation-reflection $S(O,\frac{1}{2},\ell)$.*

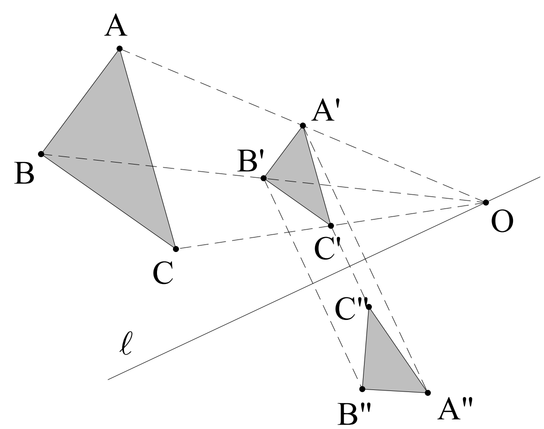


Figure 3.1.11: Transforming a triangle by $S(O,\frac{1}{2},\ell)$

Example 3.1.11. *A dilation could be a stretch and a negative value of k puts the image on the opposite side of the center O. In* Figure 3.1.12*, triangle $A''B''C''$ is the image of triangle ABC under the dilation-reflection $S(O,-2,\ell)$.*

Definition 3.1.11. *Let O be a fixed point of the plane, k a given nonzero number and θ a given sensed angle. A* dilation-rotation*, $H(O,k,\theta)$, is the product $R(O,\theta)H(O,k)$. Point O is called the* center*, k the* ratio *and θ the* angle *of the dilation-rotation. (Some call this transformation a* homology*, but this term is used in other ways in mathematics and may cause some confusion if used here.)*

Example 3.1.12. *In* Figure 3.1.13*, triangle ABC is mapped onto triangle $A'B'C'$ by the dilation $H(O,-\frac{3}{2})$. Then triangle $A'B'C'$ is mapped onto triangle $A''B''C''$ by the rotation $R(O,90°)$. The end result of the mapping of triangle ABC onto triangle $A''B''C''$ by the dilation-rotation, $H(O,-\frac{3}{2},90°)$.*

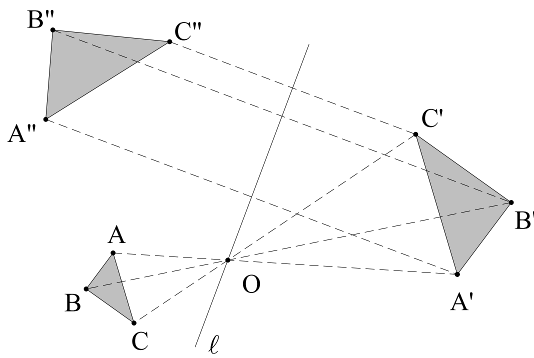


Figure 3.1.12: Transforming a triangle by $S(O, -2, \ell)$

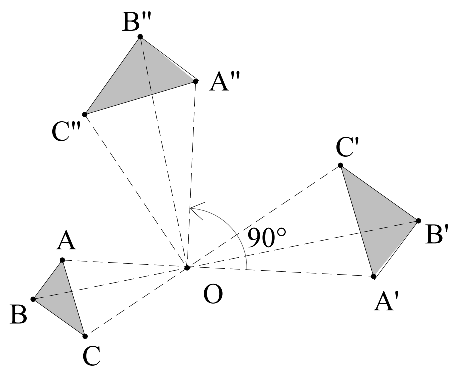


Figure 3.1.13: Transforming a triangle by $H(O, -\frac{3}{2}, 90°)$

The following theorems give some relationships among the transformations we have discussed.

Theorem 3.1.2. *If n is an integer, then $R(O,(2n+1)180°) = R(O) = H(O,-1)$.*

Theorem 3.1.3. *If n is an integer, then*

(i) $H(O,k,n \cdot 360°) = H(O,k)$.

(ii) $H(O,k,(2n+1)180°) = H(O,-k)$.

Theorem 3.1.4. $T(BC)T(AB) = T(AB)T(BC) = T(AC)$.

Theorem 3.1.5. $R(O,\theta_2)R(O,\theta_1) = R(O,\theta_1)R(O,\theta_2) = R(O,\theta_1+\theta_2)$.

Theorem 3.1.6. $R(O,\theta)H(O,k) = H(O,k)R(O,\theta) = H(O,k,\theta)$.

Theorem 3.1.7. *If $\overline{AB}$ is on ℓ, then $R(\ell)T(AB) = T(AB)R(\ell) = G(\ell, AB)$.*

Theorem 3.1.8. *If O is on ℓ, then $R(\ell)H(O,k) = H(O,k)R(\ell) = S(O,k,\ell)$.*

Since our transformations are one-to-one, onto mappings, they all have inverses. The following theorem will summarize these results.

Theorem 3.1.9. *The following all hold:*

(i) $[T(AB)]^{-1} = T(BA)$

(ii) $[R(O,\theta)]^{-1} = R(O,-\theta)$

(iii) $[R(\ell)]^{-1} = R(\ell)$

(iv) $[R(O)]^{-1} = R(O)$

(v) $[H(O,k)]^{-1} = H(O,\frac{1}{k})$

(vi) $[G(\ell,AB)]^{-1} = G(\ell,BA)$

(vii) $[H(O,k,\theta)]^{-1} = H(O,\frac{1}{k},-\theta)$

(viii) $[S(O,k,\ell)]^{-1} = S(O,\frac{1}{k},\ell)$.

We close with a final definition and comment.

Definition 3.1.12. *A transformation that is its own inverse is called* involutoric.

From the above theorem we see that $R(\ell)$ and $R(O)$ are the only two involutoric transformations in our collection.

Transformations of the plane should not be new. In algebra, trigonometry, precalculus and often calculus, transformations of the plane were used to simplify problems. Translations of axes was used to simplify graphing of the conic sections. Rotation and translation of axes was also used to help identify and graph general quadratics of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Brief discussions of these can be found in Appendix A for those who wish to refresh their memory.

Exercises

3.1.1. *The $xy-$plane is mapped onto itself by the translation $T(AB)$ where $A = (1,1)$ and $B = (3,-1)$. What are the coordinates of the images of the following points under this transformation?*
(a) $(2,0)$ (b) $(0,5)$ (c) $(3,7)$ (d) $(-5,4)$ (e) $(-3,-2)$.

3.1.2. *The $xy-$plane is mapped onto itself by the rotation $R(O,60^\circ)$ where O is the origin $(0,0)$. What are the coordinates of the images of the following points under this transformation?*
(a) $(1,1)$ (b) $(-2,1)$ (c) $(4,0)$ (d) $(0,3)$ (e) $(4,-5)$.

3.1.3. *The xy−plane is mapped onto itself by the reflection $R(O)$ where O is the point $(1,-2)$. What are the coordinates of the images of the following points under this transformation?*
(*a*) $(1,1)$ (*b*) $(-2,0)$ (*c*) $(0,0)$ (*d*) $(0,5)$ (*e*) $(-3,1)$.

3.1.4. *The xy−plane is mapped onto itself by the homothety (dilation) $H(O,\frac{1}{2})$ where O is the point $(-2,1)$. What are the coordinates of the images of the following points under this transformation?*
(*a*) $(0,0)$ (*b*) $(1,3)$ (*c*) $(4,3)$ (*d*) $(1,-1)$ (*e*) $(-6,-5)$.

3.1.5. *The xy−plane is mapped onto itself by the reflection $R(\ell)$ where ℓ is the line with equation $x-2y=4$. What are the coordinates of the images of the following points under this transformation?*
(*a*) $(0,0)$ (*b*) $(1,1)$ (*c*) $(-3,-2)$ (*d*) $(4,0)$ (*e*) $(6,-1)$.

3.1.6. *Let $P'(x_1,y_1)$ be the image of the point $P(x_0,y_0)$ under a reflection through the line given by $ax+by+c=0$. Derive the algebraic expressions for x_1 and y_1 in terms of a,b,c,x_0,y_0.*

3.1.7. *If AB is carried into $A'B'$ by a rotation, locate the center of the rotation. Be sure to consider all possible cases.*

3.1.8. *Let $ABCD$ be a square with center O. Show that*

$$R(B,90^\circ)R(C,90^\circ)=R(O).$$

3.1.9. *Show that $R(O_2)R(O_1)=T(2O_1O_2)$.*

3.1.10. *In part (b), the O' is the one from part (a).*

- *(a) Show that $T(AB)R(O)$ is a reflection in point O' such that $\overline{OO'}$ is equal and parallel to $\dfrac{\overline{AB}}{2}$.*
- *(b) Show that $T(OO')R(O)=R(M)$, where M is the midpoint of OO'.*

3.1.11. *Show that $R(O_3)R(O_2)R(O_1)$ is a reflection in point O such that $\overline{OO_3}$ is equal and parallel to $\overline{O_1O_2}$.*

3.1.12. A review. *Let O,P,M,N, in a rectangular cartesian coordinate system, be the points $(0,0),(1,1),(1,0),(2,0)$ respectively, and let ℓ denote the $x-$axis. Find the coordinates of the point P' obtained from the point P by the following transformations: (a) $T(OM)$, (b) $R(O,90^\circ)$, (c) $R(\ell)$, (d) $R(M)$, (e) $R(O)$, (f) $H(O,2)$, (g) $H(N,-2)$, (h) $H(M,\frac{1}{2})$, (i) $G(\ell,MN)$, (j) $S(O,2,\ell)$, (k) $H(O,2,90^\circ)$, (l) $H(N,2,45^\circ)$*

3.2 Homothety Applications

Notation. By the symbol $O(r)$ we will mean the circle with center O and radius r.

Definition 3.2.1. *Let $A(a)$ and $B(b)$ be two nonconcentric circles and let I and E divide $\overline{AB}$ internally and externally in the ratio $\frac{a}{b}$. Then I and E are called the* internal *and* external centers of similitude *of the two circles.*

Figure 3.2.1, below, shows the *internal* and *external centers of similitude* of two nonintersecting circles and two intersecting circles.

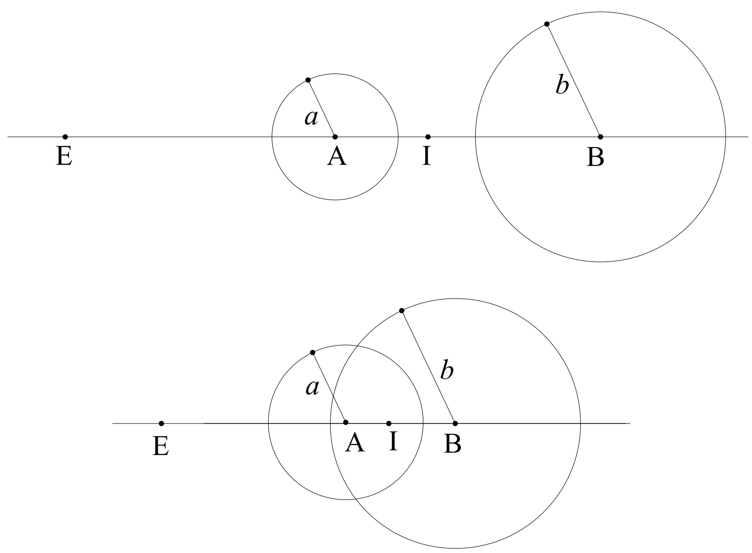


Figure 3.2.1: Internal and External Centers of Similitude

Theorem 3.2.1. *Any two nonconcentric circles $A(a)$ and $B(b)$ with internal and external centers of similitude I and E are homothetic to each other with the homotheties $H(I, -b/a)$ and $H(E, b/a)$.*

Proof. Figure 3.2.2, shows two of the possibilities for the circles. The proof is the same for all cases.

Let P be any point on $A(a)$ not collinear with A and B. Let $P'BP''$ be the diameter of $B(b)$ parallel to AP, where $\overline{BP'}$ has the same direction as $\overline{AP}$. Let PP' cut AB in E' and $P''P$ cut AB in I'. Now $\triangle E'PA \sim \triangle E'P'B$, since $AP \parallel BP'$. Thus, $\overline{E'B}/\overline{E'A} = \overline{BP'}/\overline{AP} = b/a$. Hence $E' = E$, the external center of similitude, and $B(b)$ is the image of $A(a)$ under the homothety

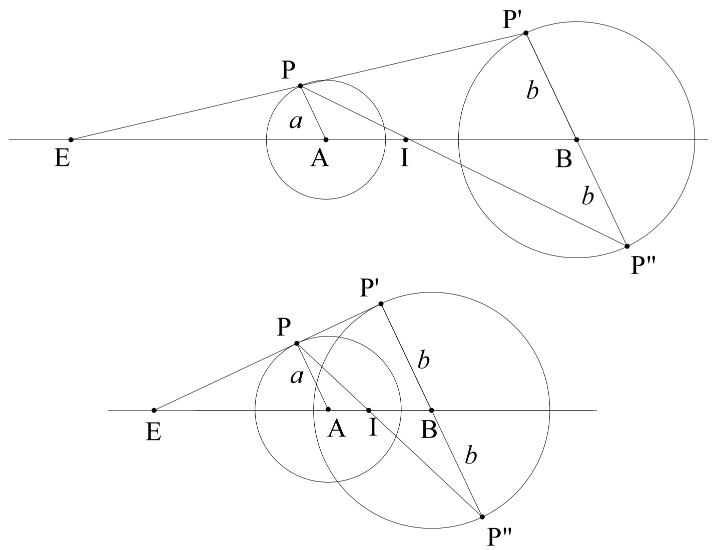


Figure 3.2.2

$H(E, b/a)$. Similarly, $\triangle I'PA \sim \triangle I'P''B$ and $\overline{I'B}/\overline{I'A} = \overline{BP''}/\overline{AP} = -b/a$. Hence $I' = I$, the internal center of similitude. Therefore, $B(b)$ is the image of $A(a)$ under the homothety $H(I, -b/a)$.

Before we proceed, we need to prove a result about the medians of a triangle and the centroid. We have previously established that the medians of a triangle are concurrent. We now would like to prove the following result that some may be familiar with.

Theorem 3.2.2. *In triangle $A_1A_2A_3$ let M_1, M_2, M_3 be the midpoints of sides A_2A_3, A_3A_1, A_1A_2 and let G be the centroid. Then $\overline{A_iG}/\overline{GM_i} = 2$, for $i = 1, 2, 3$.*

Proof. In Figure 3.2.3 below, let M_1, M_2, M_3 be the midpoints of sides A_2A_3, A_3A_1, A_1A_2 of triangle $A_1A_2A_3$. Furthermore, let N_1, N_2, N_3 be the midpoints of A_1G, A_2G, A_3G. Join the points N_1, N_2, M_1, M_2 to form the quadrilateral $N_1N_2M_1M_2$. Since N_1 and N_2 are midpoints of sides A_1G and A_2G of triangle A_1GA_2, N_1N_2 is parallel to and equal to one-half A_1A_2. Similarly, M_1M_2 is parallel to and equal to one-half A_1A_2. Now since, if two nonconsecutive sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram, $N_1N_2M_1M_2$ is a parallelogram. The diagonals of a parallelogram bisect each other, so $N_1G = GM_1$ and $N_2G = GM_2$. Thus $A_1N_1 = N_1G = GM_1$ and $A_2N_2 = N_2G = GM_2$ which implies $A_1G = 2GM_1$ and $A_2G = 2GM_2$. Using quadrilateral $N_2N_3M_2M_3$, by a similar argument we can show $A_3G = 2GM_3$.

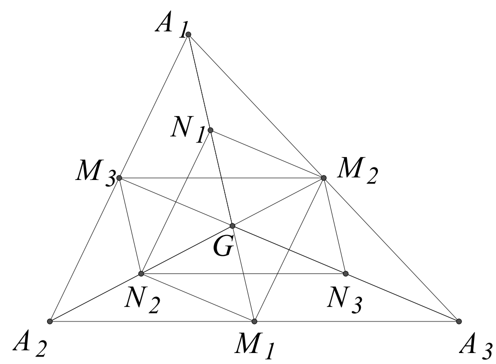


Figure 3.2.3

Theorem 3.2.3. *The orthocenter H, the circumcenter O, and the centroid G of a triangle $A_1A_2A_3$ are collinear and $\overline{HG} = 2\overline{GO}$.*

Proof.

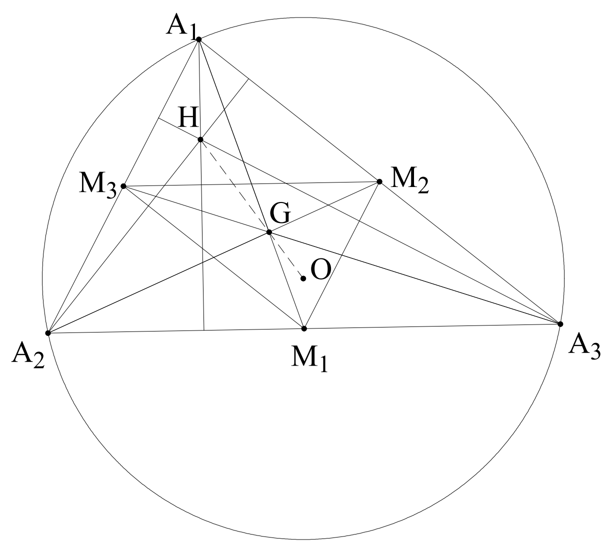


Figure 3.2.4

In Figure 3.2.4 above, let M_1, M_2, M_3 be the midpoints of sides A_2A_3, A_3A_1, A_1A_2 of triangle $A_1A_2A_3$. Since $\overline{A_iG}/\overline{GM_i} = 2,$ for $i = 1, 2, 3$, triangle $M_1M_2M_3$ is carried into triangle $A_1A_2A_3$ by the homothety $H(G, -2)$. Therefore, O which is the orthocenter of triangle $M_1M_2M_3$, maps into the orthocenter H of triangle $A_1A_2A_3$. It the follows that H, G, O are collinear and that $\overline{HG} = 2\overline{GO}$.

Definition 3.2.2. *The line of collinearity of the orthocenter, circumcenter and centroid of a triangle is called the* Euler line *of the triangle.*

Theorem 3.2.4. *In triangle $A_1A_2A_3$ let M_1, M_2, M_3 be the midpoints of the sides A_2A_3, A_3A_1, A_1A_2, let H_1, H_2, H_3 be the feet of the altitudes on these three sides, let N_1, N_2, N_3 be the midpoints of the segments A_1H, A_2H, A_3H, where H is the orthocenter of the triangle. Then the nine points M_1, M_2, M_3, H_1, H_2, H_3, N_1, N_2, N_3 lie on a circle whose center N is the midpoint of the segment joining the orthocenter H to the circumcenter O of the triangle, and whose radius is half the circumradius of the triangle.*

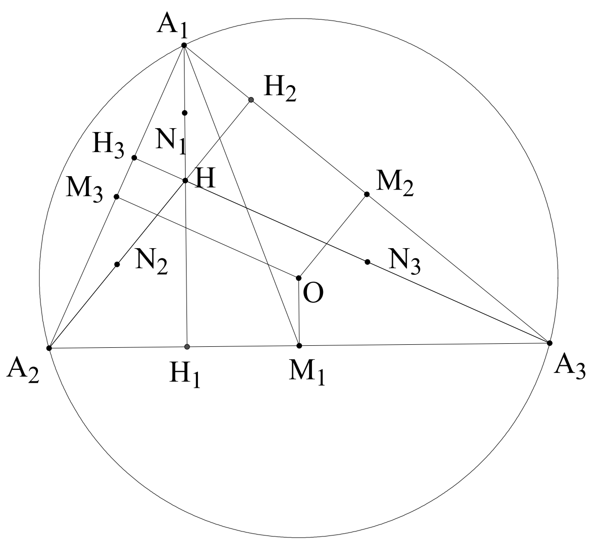


Figure 3.2.5

Proof. In Figure 3.2.5 above, we extend A_1H_1 to a point S_1, A_2H_2 to a point S_2, and A_3H_3 to a point S_3 on the circumcircle, as shown in Figure 3.2.6 below.
$\angle A_2A_3H_3 = \angle A_2A_1S_1$ since they have corresponding sides that are perpendicular. Furthermore, $\angle A_2A_1S_1 = \angle A_2A_3S_1$ since they subtend the same arc on the circumcircle. Draw S_1A_3. Thus, $\triangle HH_1A_3 \cong \triangle S_1H_1A_3$ (both are also right triangles) and hence H_1 is the midpoint of HS_1, since $HH_1 \cong H_1S_1$. Similarly, H_2 is the midpoint of HS_2 and H_3 is the midpoint of HS_3. Draw circumdiameters A_1T_1, A_2T_2, A_3T_3. Then $\angle A_1A_2T_1 = 90°$, since it is inscribed in a semicircle. Thus, T_1A_2 is parallel to A_3H_3 since both lines are perpendicular to A_1A_2. Similarly, $\angle A_1A_3T_1 = 90°$ and T_1A_3 is parallel to A_2H_2. Therefore, $HA_3T_1A_2$ is a parallelogram and since the diagonals of a parallelogram bisect each other, HT_1 and A_2A_3 bisect each other. That is, M_1 is the midpoint of HT_1. By a similar argument, M_2 is the midpoint of HT_2 and M_3 is the midpoint

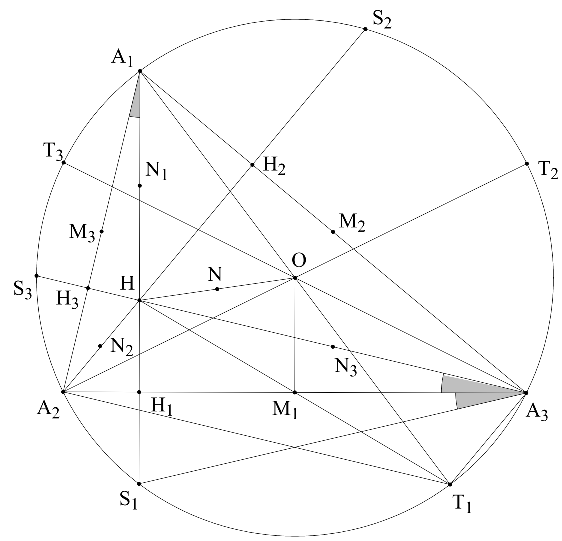


Figure 3.2.6

of HT_3.

It now follows that the homothety $H(H, \frac{1}{2})$ carries $A_1, A_2, A_3, S_1, S_2, S_3, T_1, T_2, T_3$ into the points $N_1, N_2, N_3, H_1, H_2, H_3, M_1, M_2, M_3$. These last nine points lie on a circle of radius one-half that of the circumcircle and with center N, the midpoint of HO. This circle is called the *nine-point circle* for triangle $A_1A_2A_3$.

Definition 3.2.3. *Let I and E be the internal and external centers of similitude of two given nonconcentric circles $A(a), B(b)$ having unequal radii. Then the circle on IE as diameter is called the* circle of similitude *of the two given circles.*

Theorem 3.2.5. *Let P be any point on a circle of similitude of two nonconcentric circles $A(a), B(b)$ having unequal radii. Then $B(b)$ is the image of $A(a)$ under the dilation-rotation (homology) $H(P, \frac{b}{a}, \angle\overline{APB})$.*

Proof. In Figure 3.2.7 below, let I and E be the internal and external centers of similitude of the two given circles. If P coincides with I or E, the theorem follows from the fact that $H(I, \frac{b}{a}, 180°) = H(I, -\frac{b}{a})$ or $H(E, \frac{b}{a}, 0°)$. If P is distinct from I and E, as in Figure 3.2.7, then PI is perpendicular to PE since $\angle IPE$ is inscribed in a semicircle. Draw PA' so that PI bisects $\angle A'PB$ internally. Then PE is the external bisector of the same angle and it follows, from Exercise 2.6.12, that $(A'B, IE) = -1$. But $(AB, IE) = \frac{\overline{AI}}{\overline{IB}} \cdot \frac{\overline{EB}}{\overline{AE}} = \left(\frac{a}{b}\right)\left(-\frac{b}{a}\right) = -1$. Thus, $A' = A$. Now, since PI and PE are the internal and external bisectors of $\angle APB$ we have that $\frac{PB}{PA} = \frac{\overline{IB}}{\overline{AI}} = \frac{b}{a}$ (by Theorem 2.2.5) and the theorem follows.

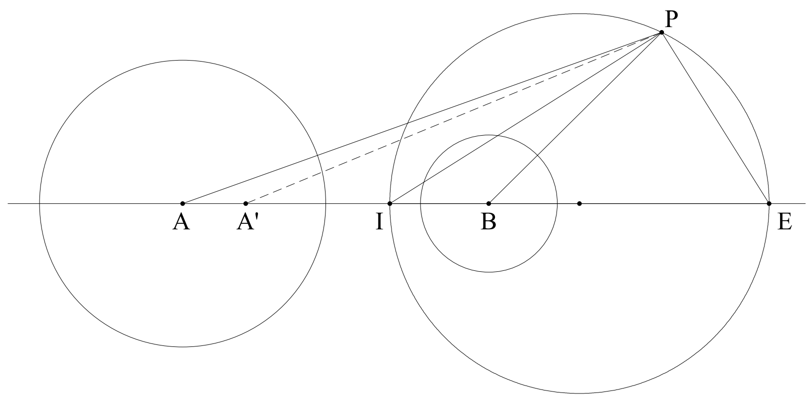


Figure 3.2.7

Corollary. *The locus of a point P moving in a plane such that the ratio of its distance from point A to its distance from point B of the plane is a positive constant $k \neq 1$ is the circle on IE as diameter, where I and E divide the segment AB internally and externally in the same ratio k.* (This circle is called the *circle of Apollonius* of points A and B for the ratio k.)

Exercises

3.2.1. *Prove that if two circles have common external tangents, these tangents pass through the external center of similitude of the two circles, and if they have common internal tangents, these pass through the internal center of similitude.*

3.2.2. *Show that the external centers of similitude of three circles with distinct centers taken in pairs are collinear.*

3.2.3. *Show that the external center of similitude of one pair of circles and the internal centers of similitude of the other two pair are collinear.*

3.2.4. *Show that any circle through the centers of two given nonconcentric circles of unequal radii is orthogonal to the circle of similitude of the two given circles.*

3.2.5. *Given two parallel lines. Using a straightedge only, divide a segment on one of the lines into six equal parts.*

3.3 Isometries

Definition 3.3.1. *An* ***isometry*** *of the plane is a map from the plane to itself which preserves distances. That is, f is an isometry if for any P and Q in the plane, we have*

$$\overline{f(P)f(Q)} = \overline{PQ}.$$

We see from our previous work that an isometry is either a *translation*, *reflection*, *rotation* or a *glide-reflection*.

Definition 3.3.2. *Two sets of points (defining a triangle, angle, or some other figure) are* ***congruent*** *if there exists an isometry which maps one set to the other.*

Definition 3.3.3. *An isometry is said to be* direct *or* proper *if it preserves the orientation of the figure. If the orientation is reversed, the isometry is said to be* opposite *or* improper.

Theorem 3.3.1. *There is a unique isometry that carries a given triangle ABC into a given congruent triangle $A'B'C'$.*

Proof. We can superimpose the plane (by sliding, or turning it over and then sliding) upon itself so that $\triangle ABC$ coincides with $\triangle A'B'C'$ to induce an isometry of the plane onto itself in which the three points A, B, C are carried into the points A', B', C'. There is only one isometry, for if P is any point in the plane, there is a unique point P' in the plane such that $P'A' = PA,\ P'B' = PB,\ P'C' = PC$.

The next theorem is a rather remarkable result in that it shows that if we have two congruent triangles located anywhere in the plane, we can map one onto the other with at most three reflections in lines.

Theorem 3.3.2. *An isometry can be expressed as the product of at most three reflections in lines.*

Note. The following is an excellent example of solving a problem by reducing it to one that is already known to be solvable.

Proof. It is sufficient to consider a noncollinear triad of points (a triangle). Let an isometry carry the triad of points A, B, C into the congruent triad A', B', C'. We need only consider four cases (Figures for these cases are given at the end of the proof):

(1) If the two triads coincide, as in Figure 3.3.1 the isometry would be the identity map, which can be represented as the result of two reflections through the same line ℓ, where ℓ is any line in the plane.(See Figure 3.3.2) Here $\triangle ABC$ is reflected through ℓ to $\triangle A'B'C'$ then $\triangle A'B'C'$ is reflected through ℓ to $\triangle PQR$.

(2) If A coincides with P and B with Q, but C and R are distinct, then the isometry is the reflection through ℓ, where ℓ is the line containing the line segment AB.

This will map C onto R and we are through.

(3) If A coincides with P, but B and Q and C and R are distinct, the reflection where ℓ is the perpendicular bisector of BQ, reduces this case to one of the two previous cases. In the Figure 3.3.4 below A coincides with P and when the reflection with ℓ as the perpendicular bisector of BQ is made, we reduce to case (1), so we need only one reflection to complete the isometry.

In the Figure 3.3.5 below A coincides with P and when the reflection with ℓ_1 as the perpendicular bisector of CR is made, we reduce to case (2), which requires one more reflection, with ℓ_2 as the axis of reflection–for a total of two reflections.

(4) The last possibility is that P, Q, R are distinct from A, B, C, and the reflection where ℓ_1 is the perpendicular bisector of BQ, reduces this case to one of the three previous cases. Figure 3.3.6 illustrates one possibility with two reflections and Figure 3.3.7 illustrates one possibility with three reflections.

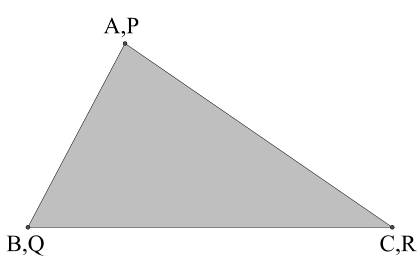


Figure 3.3.1: Triads Coincide

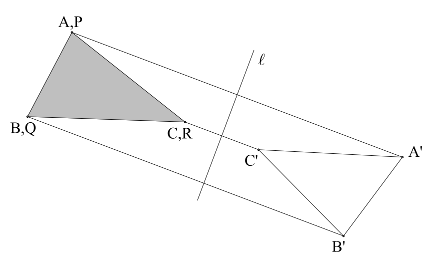


Figure 3.3.2: A Reflection

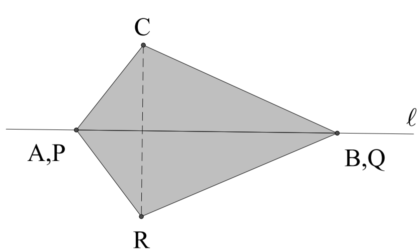


Figure 3.3.3: Side Coincides

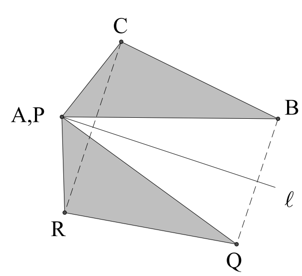


Figure 3.3.4: One Fixed Point

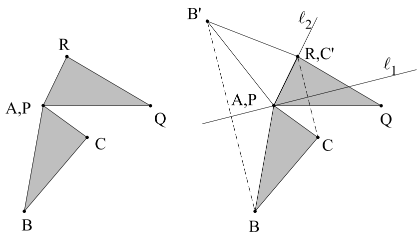


Figure 3.3.5: Fixed Point Variation

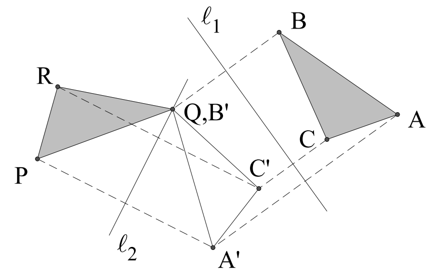


Figure 3.3.6: No Fixed Points

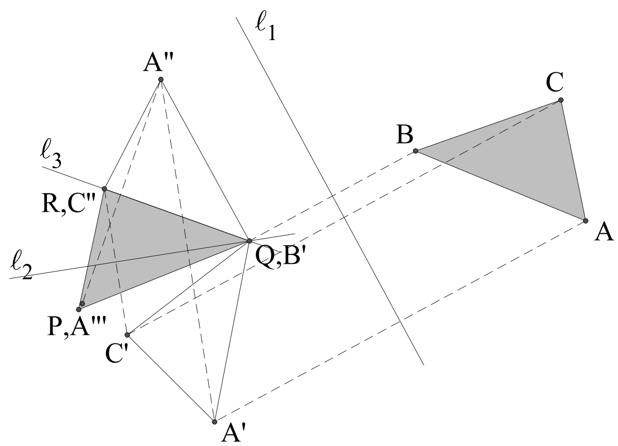


Figure 3.3.7: Three Reflections

In each of the cases illustrated in Figures 3.3.1 through 3.3.7, the isometry is expressible as a product of no more than three reflections in lines. Notice that in all of the cases our objective was to reduce the problem at hand to a previously solved problem.

We note from the above cases that if there are zero or two reflections in lines required, the isometry is a *direct isometry.* But if one or three reflections are required, the isometry is an *opposite isometry.*

Theorem 3.3.3. *An isometry with an invariant (fixed) point can be represented as a product of at most two reflections in lines.*

This is almost a corollary to Theorem 3.3.2, for suppose A is the invariant point of the isometry. Let B and C be any two points not collinear with A. Then the triangle ABC is mapped to the triangle $A'B'C'$ where A' coincides with A. The theorem now follows from the first three cases of Theorem 3.3.2.

Theorem 3.3.4. *Let ℓ_1 and ℓ_2 be any two lines of the plane intersecting in a point O, and let θ be the directed angle from ℓ_1 to ℓ_2, then $R(\ell_2)R(\ell_1) = R(O, 2\theta)$. Conversely, a rotation $R(O, 2\theta)$ can be factored into the product $R(\ell_2)R(\ell_1)$ of reflections in two lines ℓ_1 and ℓ_2 through O, where either line may be arbitrarily chosen through O and then the other such that the directed angle from ℓ_1 to ℓ_2 is equal to θ.*

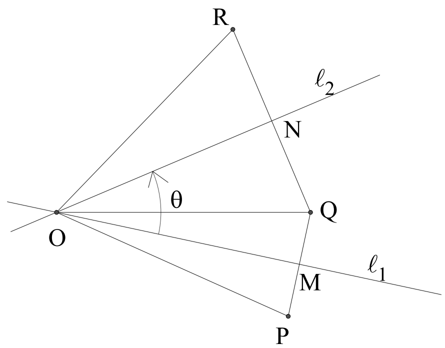


Figure 3.3.8: Theorem 3.3.4

Theorem 3.3.5. *Let ℓ_1 and ℓ_2 be any two parallel (or coincident) lines of the plane, and let $\overline{A_1A_2}$ be the directed distance from line ℓ_1 to line ℓ_2, then $R(\ell_2)R(\ell_1) = T(2A_1A_2)$. Conversely, a translation $T(2A_1A_2)$ can be factored into the product $R(\ell_2)R(\ell_1)$ of reflections in two lines ℓ_1 and ℓ_2 perpendicular to A_1A_2, where either line may be arbitrarily chosen perpendicular to A_1A_2 and then the other such that the directed distance from ℓ_1 to ℓ_2 is equal to $\overline{A_1A_2}$.*

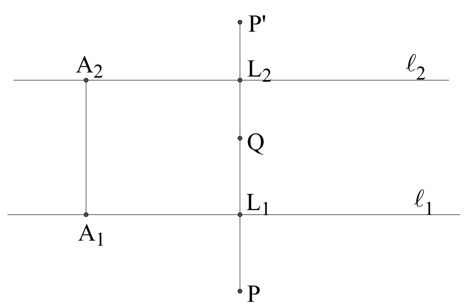


Figure 3.3.9: Theorem 3.3.5

Proof. In Figure 3.3.9, let L_1 and L_2 be the points where PP' crosses ℓ_1 and ℓ_2, let Q be the image of P reflected through ℓ_1 and let P' be the image of the reflection of Q through ℓ_2. We see that

$$\overline{PP'} = \overline{PQ} + \overline{QP'} = 2\overline{L_1Q} + 2\overline{QL_2} = 2(\overline{L_1Q} + \overline{QL_2}) = 2\overline{A_1A_2}.$$

Since PP' and A_1A_2 are both perpendicular to ℓ_1, they are parallel and the result follows. The second part follows immediately from Figure 3.3.9.

Theorem 3.3.6. *Any direct isometry is either a translation or a rotation.*

Theorem 3.3.7. $R(O)R(\ell) = G(m, 2MO)$, *where m is the line through O perpendicular to ℓ and cutting ℓ in M.*

Theorem 3.3.8. *An opposite isometry T is either a reflection in a line or a glide-reflection.*

Theorem 3.3.9. *A product of three reflections in lines is either a reflection in a line or a glide-reflection.*

Exercises

3.3.1. *Prove Theorem 3.3.4.*

3.3.2. *Prove Theorem 3.3.6.*

3.3.3. *Prove Theorem 3.3.8.*

3.3.4. *Show that $R(\ell)T(AB)$ is a glide-reflection whose axis is a line m parallel to ℓ at a distance equal to one-half the projection of $\overline{BA}$ on a line perpendicular to ℓ, and whose vector is the projection of $\overline{AB}$ on ℓ.*

3.3.5. *Show that $T(BA)R(O,\theta)T(AB) = R(O',\theta)$, where $\overline{O'O}$ is equal and parallel to $\overline{AB}$. (Draw the figure illustrating this transformation performed on a point P.)*

3.3.6. *In the library basement among some old books you find a document showing a treasure was buried on a deserted island that contained a pyramidal rock P and three easily identifiable trees A, B, and C. The map for locating the treasure showed the relative positions of the rock and trees and gave the following directions: Start at P and go to the point P_1, which is the image of P under the half-turn $R(A)$. Continue to P_2, which is the image of P_1 under $R(B)$. Then go to P_3, which is the image of P_2 under $R(C)$. Continue finding the images under $R(A), R(B)$, and $R(C)$ until you reach P_6, where the treasure is buried.*

You immediately get a group of friends to go search for the treasure. However, when you arrive on the island you find that a hurricane wiped the island clean except for the rock. While you and others were bemoaning the fact that the only possible way to find the treasure would be to dig up the whole island, which was quite large, the mathematician in the group was scratching on the sandy beach with a stick. She soon declared, "We can find the treasure. All we have to do is pick any three points on the island to represent the missing trees."

Was she correct? Illustrate her solution.

3.4 Inversion

The inversion transformation has a murky history. It is not clear when, or by whom, the inversion transformation was first studied. We do know that one of the basic ideas of inversion was known to Apollonius (*ca.* 262 - *ca.* 190 BCE). This is that the inverse of a straight line or circle is a straight line or circle.

Definition 3.4.1. *If point P is not the center O of circle $O(r)$, the* inverse *of P in, or with respect to, circle $O(r)$ is the point P' lying on the line OP such that $(\overline{OP})(\overline{OP'}) = r^2$. Circle $O(r)$ is called the* circle of inversion*, point O the* center of inversion*, r the* radius of inversion*, and r^2 the* power of inversion*. The symbol $I(O, k)$ will be used to denote the inversion with center O and power $k > 0$.*

It is clear from the definition that if P is inside the circle, then P' must be outside the circle, and conversely. Also the boundary points of the circle of inversion are all fixed (or invariant) points of the transformation. These statements follow from the product in the definition, $(\overline{OP})(\overline{OP'}) = r^2$.

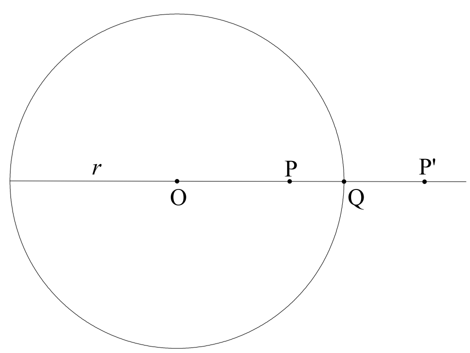


Figure 3.4.1: The Inverse of a Point

In Figure 3.4.1, P' is the inverse of P, and conversely. Also Q is its own inverse; that is, Q is a fixed point for the inversion in $O(r)$.

We can see then that the inversion transformation maps all points (other than O) inside the circle of inversion to points outside the circle, and all points outside the circle are mapped to points inside the circle. And, as mentioned before, all points on the circle are invariant. In order to not have to constantly mention that O cannot be considered as a point to be mapped, we make the following convention: for the inversion transformation we consider the plane to be the ordinary Euclidean plane with one ideal point added. The ideal point will be considered to be the image (inverse) of O under the inversion transformation.

Z will be used to designate the ideal point. We call this plane the *inversive plane.*

We summarize the comments made before Figure 3.4.1 as a theorem.

Theorem 3.4.1. *Inversion is an involutoric transformation of the inversive plane onto itself which maps the interior of the circle of inversion onto the exterior of the circle of inversion, the exterior of the circle of inversion onto the interior of the circle of inversion, and each point on the circle of inversion onto itself.*

Since an involutoric transformation is a transformation that is its own inverse, it is clear then that the inversion transformation maps the exterior of the circle onto the interior of the circle.

Theorem 3.4.2. *A point D outside the circle of inversion and the point C where the points of contact of the tangents from D to the circle of inversion cuts the diametral line OD are inverse points*

Proof. Consider Figure 3.4.2. Since $\triangle OCT \sim \triangle OTD$, we have that

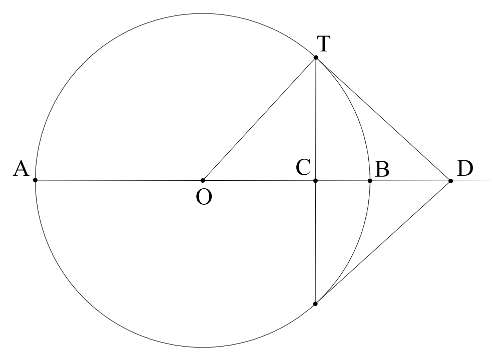


Figure 3.4.2

$(\overline{OC})(\overline{OD}) = (\overline{OT})^2 = r^2$, C and D are, by definition, inverses. □

The above theorem not only gives us a method of constructing the inverse of a given point, it also shows that C and D are harmonic conjugates with respect to A and B. That is, $(AB, CD) = -1$. Note that the method of constructing inverse points is one of the methods of construction we had for harmonic conjugates. This observation can be formalized in the following theorem.

Theorem 3.4.3. *If C, D are inverse points with respect to circle $O(r)$, then $(AB, CD) = -1$, where AB is the diameter of $O(r)$ through C and D; conversely, if $(AB, CD) = -1$, where AB is the diameter of $O(r)$, then C and D are inverse points with respect to circle $O(r)$.*

Proof. In Figure 3.4.2 above, we have that $(\overline{OC})(\overline{OD}) = r^2 = (\overline{OB})^2$ if and only if $(AB, CD) = -1$. □

Note that $(AB, OZ) = -1$. That is, the ideal point Z is the image of O under an inversion $O(r)$ and the harmonic conjugate of the midpoint of a segment is the ideal point on that line.

Recall that we introduced the term "circle" to represent a circle or a straight line. We make use of this term again.

Theorem 3.4.4. *A "circle" orthogonal to the circle of inversion inverts into itself.*

Proof. The result is obvious if the "circle" is a straight line, and the proof of the theorem from the section on orthogonal circles that states

> **Theorem 2.7.2** *If two circles are orthogonal, then any diameter of one which intersects the other is cut harmonically by the other; conversely, if a diameter of one circle is cut harmonically by a second circle, then the two circles are orthogonal.*

takes care of the case where the "circle" is a circle. □

An alternative statement for Theorem 2.7.2 is given below.

Theorem 3.4.5. *If C, D are inverse points with respect to circle $O(r)$, then any circle through C and D cuts circle $O(r)$ orthogonally; conversely, if a diameter of circle $O(r)$ cuts a circle orthogonal to $O(r)$ in C and D, then C and D are inverse points with respect to $O(r)$*

Theorem 3.4.6. *If two intersecting circles are each orthogonal to a third circle, then the points of intersection of the two circles are inverse points with respect to the third circle.*

Proof. Consider Figure 3.4.3.
Let the two circles with centers O' and O'' intersect in points C and D and let O be the third circle, as in Figure 3.4.3. Draw OC to cut the two circles again in D' and D''. Then by Theorem 3.4.5, D' and D'' are each the inverse of C with respect to the third circle, O. Thus, it follows that $D' = D'' = D$ and C and D are inverse points with respect to the third circle. □

Theorem 3.4.7. $I(O, k_2)I(O, k_1) = H(O, k_2/k_1)$.

Proof. Let P be any point other than O. The inversion $I(O, k_1)$ will carry P into some point P', which is collinear with O and P. Then $I(O, k_2)$ will carry

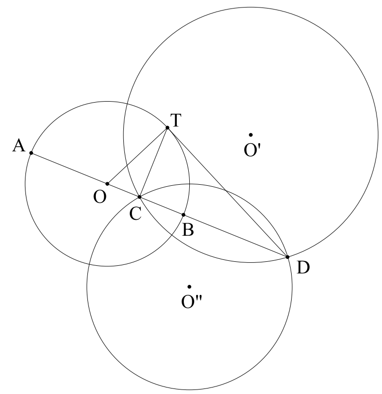


Figure 3.4.3

P' into some point P'' which is collinear with O and P'. Thus, P, P', P'', and O are all collinear. Furthermore we have that

$$(\overline{OP})(\overline{OP'}) = k_1 \text{ and } (\overline{OP'})(\overline{OP''}) = k_2.$$

Hence

$$\frac{\overline{OP''}}{\overline{OP}} = \frac{k_2}{k_1} \text{ and thus, } \overline{OP''} = \frac{k_2}{k_1}\overline{OP}.$$

But this is the homothetic transformation $H(O, k_2/k_1)$. On the other hand, if $P = O$, then $P' = Z, P'' = O$ and the result still holds. $\square$

We will now examine the images of lines and circles under inversions. We begin with a special case of Theorem 3.4.4, since a diameter line for a circle is considered a "circle" orthogonal to the given circle.

Theorem 3.4.8. *The inverse of a straight line ℓ passing through the center O of inversion is the line ℓ itself.*

Theorem 3.4.9. *The inverse of a straight line ℓ not passing through the center O of inversion is a circle C passing through O and having its diameter through O perpendicular to ℓ.*

Proof. Let point A(see Figure 3.4.4) be the foot of the perpendicular dropped from O on ℓ. Let P be any other ordinary point on ℓ and let A' and P' be the inverses of A and P. Then $(\overline{OA})(\overline{OA'}) = (\overline{OP})(\overline{OP'})$ whence $\frac{\overline{OP'}}{\overline{OA'}} = \frac{\overline{OA}}{\overline{OP}}$ and $\triangle OP'A' \sim \triangle OAP$. Therefore $\angle OP'A' = \angle OAP = 90°$. It then follows that P' lies on the circle C having OA' as diameter. Conversely, if P' is any point on circle C other than O or A' let OP' cut line ℓ at P. Then P' must be the inverse of P. Note that point O on circle C corresponds to the

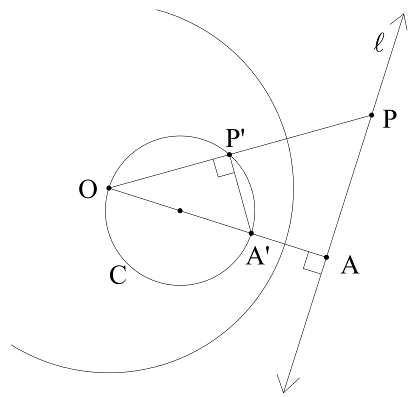


Figure 3.4.4: Inverse of a Line

point Z at infinity on ℓ. □

Theorem 3.4.10. *The inverse of a circle C passing through the center O of inversion is a straight line ℓ not passing through O and perpendicular to the diameter of C through O.*

Proof. Let A (see Figure 3.4.5) be the point on C diametrically opposite O, and let P be any point on circle C other than O or A. Let A', P' be inverses of A, P. Then $(\overline{OA})(\overline{OA'}) = (\overline{OP})(\overline{OP'})$ whence $\frac{\overline{OP'}}{\overline{OA'}} = \frac{\overline{OA}}{\overline{OP}}$ and $\triangle OP'A' \sim \triangle OAP$. Therefore $\angle OA'P' = \angle OPA = 90°$. It follows then that P' lies on the line through A' and perpendicular to OA. Conversely, if P' is an ordinary point on line ℓ other than A', let OP' cut circle C in P. Then, by the above, P' must be the inverse of P. Note that the point at infinity on ℓ corresponds to the point O on circle C. □

Theorem 3.4.11. *The inverse of a circle C not passing through the center O of inversion is a circle C' not passing through O and homothetic to circle C with O as the center of the homothety.*

Proof. Let P (see Figure 3.4.6) be any point on circle C. Let P' be the inverse of P and let OP meet circle C again in Q (Q and P will coincide if OP is tangent to C). Let r^2 be the power of inversion and let p be the power of the point O with respect to circle C. Then $(\overline{OP})(\overline{OP'}) = r^2$ and $(\overline{OP})(\overline{OQ}) = p$, whence $\frac{\overline{OP'}}{\overline{OQ}} = \frac{r^2}{p}$, which is a constant. Therefore it follows that P' describes the map of the locus of Q under the homothety $H(O, r^2/p)$. That is, P describes a circle C' homothetic to circle C and having O as center of homothety. Since C

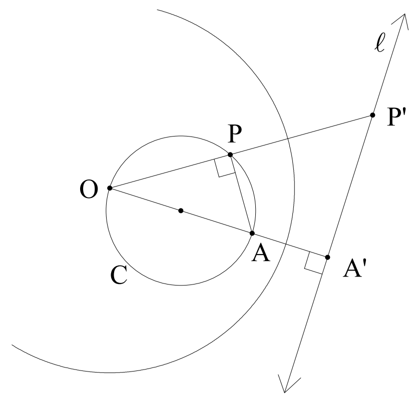


Figure 3.4.5: Inverse of a Circle Through Center of Inversion

does not pass through O, circle C' does not pass through O. $\square$

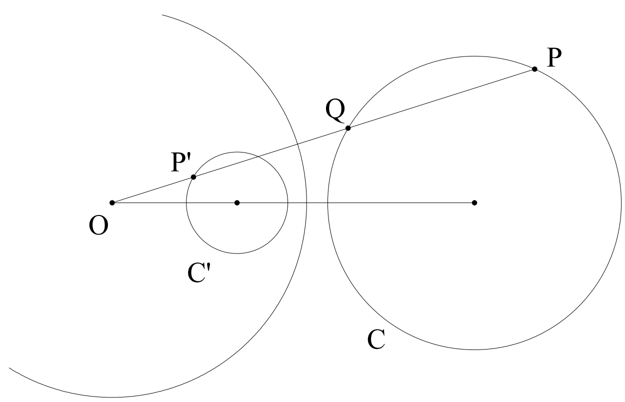


Figure 3.4.6: Homothetic Circles

Note: In the above figure, the centers of the two circles C and C' are represented. **It should be noted that they are not images of each other under the inversion.**

We state the following properties of inversion without proof.

Lemma 3.4.1. *Let C' be the inverse of "circle" C, and let P and P' be a pair of corresponding points (which may be coincident), under an inversion of center O, on C and C' respectively, then the tangents to C and C' at P and P' are reflections of one another in the perpendicular to OP through the midpoint of*

PP'.

NOTE. Recall that if a "circle" is a straight line, then it is its own tangent at all points.

There are two configurations for Lemma 3.4.1 that we must consider (see Figure 3.4.7).

The proof involves showing that (in Figure 3.4.7 a) $\angle APP' = \angle AP'P$, so that triangle APP' is isosceles. Likewise in Figure 3.4.7 b, show that triangle BPP' is isosceles.

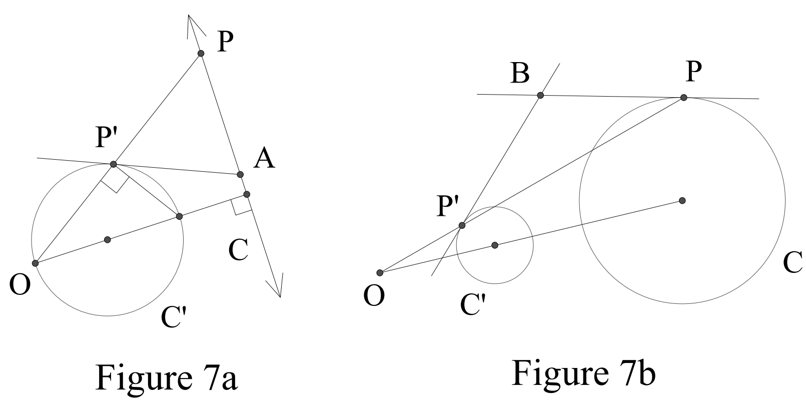


Figure 3.4.7: Reflections of Tangents

Theorem 3.4.12. *A directed angle of intersection of two "circles" is unaltered in magnitude but reversed in sense by inversion.*

Corollary 3.4.1. (1) *If two "circles" are tangent, their inverses are tangent.* (2) *If two "circles" are orthogonal, their inverses are orthogonal.*

Theorem 3.4.13. *If P, P' and Q, Q' are pairs of inverse points with respect to circle $O(r)$, then $P'Q' = \dfrac{(PQ)r^2}{(OP)(OQ)}$.*

Proof. We must consider two cases. First, suppose O, P, Q are collinear.

$$\begin{aligned}
(\overline{OP})(\overline{OP'}) &= (\overline{OQ})(\overline{OQ'}) \\
(\overline{OQ} + \overline{QP})\overline{OP'} &= \overline{OQ}(\overline{OP'} + \overline{P'Q'}) \\
\overline{OQOP'} + \overline{QPOP'} &= \overline{OQOP'} + \overline{OQP'Q'} \\
(\overline{QP})(\overline{OP'}) &= (\overline{OQ})(\overline{P'Q'}) \\
\overline{P'Q'} &= \frac{(\overline{QP})(\overline{OP'})}{\overline{OQ}}
\end{aligned}$$

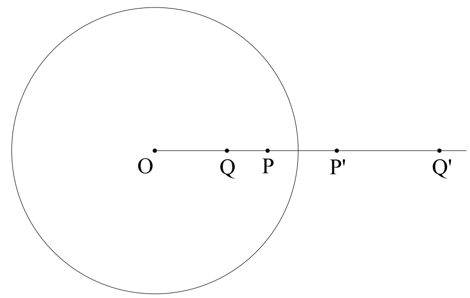


Figure 3.4.8

If we now multiply the right hand side of the above by $\dfrac{\overline{OP}}{\overline{OP}}$ we have

$$\overline{P'Q'} = \frac{(\overline{QP})(\overline{OP'})(\overline{OP})}{(\overline{OP})(\overline{OQ})} = \frac{(\overline{QP})r^2}{(\overline{OP})(\overline{OQ})}.$$

Since direction is not critical here, we can write this last equation as

$$P'Q' = \frac{(PQ)r^2}{(OP)(OQ)}.$$

Now suppose O, P, Q are not collinear. See Figure 3.4.9 below.

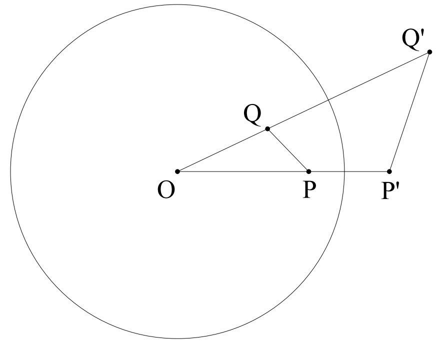


Figure 3.4.9

Since $r^2 = (\overline{OP})(\overline{OP'}) = (\overline{OQ})(\overline{OQ'})$, triangle OPQ is similar to triangle $OQ'P'$, and therefore, $\dfrac{P'Q'}{PQ} = \dfrac{OQ'}{OP}$. If we now multiply the right side of this equation by $\dfrac{\overline{OQ}}{\overline{OQ}}$, we have $\dfrac{P'Q'}{PQ} = \dfrac{(OQ')(OQ)}{(OP)(OQ)}$. Hence,

$$P'Q' = \frac{(PQ)r^2}{(OP)(OQ)}.$$

Exercises

3.4.1. *If a point P is 4 units from the center of inversion and the radius of the circle of inversion is 5 units, how far is the image point P' from the center of inversion?*

3.4.2. *If a point P is 3 units from the center of inversion and its image P' is $\frac{4}{3}$ units from the center of inversion, what is the radius of the circle of inversion?*

3.4.3. *Describe the location of the inverse of any point inside the circle of inversion.*

3.4.4. *If $I(O,k)P = P'$, what is the result of the following product*

$$I(O,k)I(O,k)I(O,k)I(O,k)P?$$

Explain your answer.

3.4.5. *If $I(O,k)P = P'$, what is the result of the following product*

$$I(O,k)I(O,k)I(O,k)P?$$

Explain your answer.

3.4.6. *Construct the image of a line not cutting the circle of inversion.*

3.4.7. *Construct the image of a line intersecting the circle of inversion.*

3.4.8. *Construct the image of a square and its diagonals under the following conditions*

(a) *with the center of inversion the center of the square and the circle of inversion inside the square.*

(b) *with the center of inversion the center of the square and the circle of inversion outside the square.*

(c) *with the center of inversion the center of the square and the square is inscribed in the circle of inversion.*

3.4.9. *Construct the image of a square and its diagonals with the center of inversion a vertex of the square.*

3.5 Applications of Inversion

In this section we will give three examples of how we can use the inversion transformation to solve some interesting problems. Without the inversion transformation these proofs can be rather difficult. The key factor is the property of inversion that maps "circles" into lines. We begin with the following problem.

Problem 3.5.1. *Let two circles Σ_1 and Σ_2 intersect in the points A and B, and let the diameters of Σ_1 and Σ_2 through B cut Σ_1 and Σ_2 in C and D. Show that the line AB passes through the center of circle BCD.*

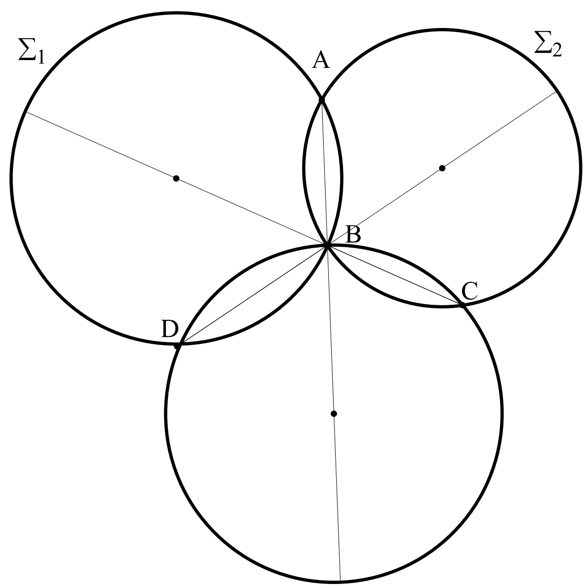


Figure 3.5.1

In Figure 3.5.1 we see that if we choose B to be the center of inversion, then all three circles will map into lines. Furthermore, lines BC and BD will map onto themselves, since lines through the center of inversion are diameters of the circle of inversion. These diameters are "circles" orthogonal to the circle of inversion and hence map onto themselves. Since BC lies on diameter of Σ_1, the "circles" BC and Σ_1 are orthogonal and orthogonal "circles" map onto orthogonal "circles". Hence the image of Σ_1 will be a line perpendicular to BC. Likewise,the image of Σ_2 will be a line perpendicular to BD.

Thus, in Figure 3.5.2 we see that the circles BCD, $ABD(\Sigma_1)$ and $ABC(\Sigma_2)$ have mapped to the straight lines $D'C', A'D', A'C'$, respectively, forming triangle $A'D'C'$ and the lines AB, CB, DB have mapped to the straight lines $A'B, C'B, D'B$, respectively. Since $C'B$ and $A'D'$ are images of orthogonal "circles", we see that line $C'B$ is perpendicular to side $A'D'$ of triangle $A'D'C'$

and therefore lies on an altitude for triangle $A'D'C'$. Similarly, $D'B$ is orthogonal to $A'C'$ as is also on an altitude for triangle $A'D'C'$. These two altitudes are concurrent at B which is then the orthocenter for triangle $A'D'C'$. Since $A'B$ passes through vertex A' and orthocenter B of triangle $A'D'C'$, it also must be on an altitude for the triangle. Hence $A'B$ must be perpendicular (orthogonal) to $D'C'$. Now if we invert the original transformation we must have that the

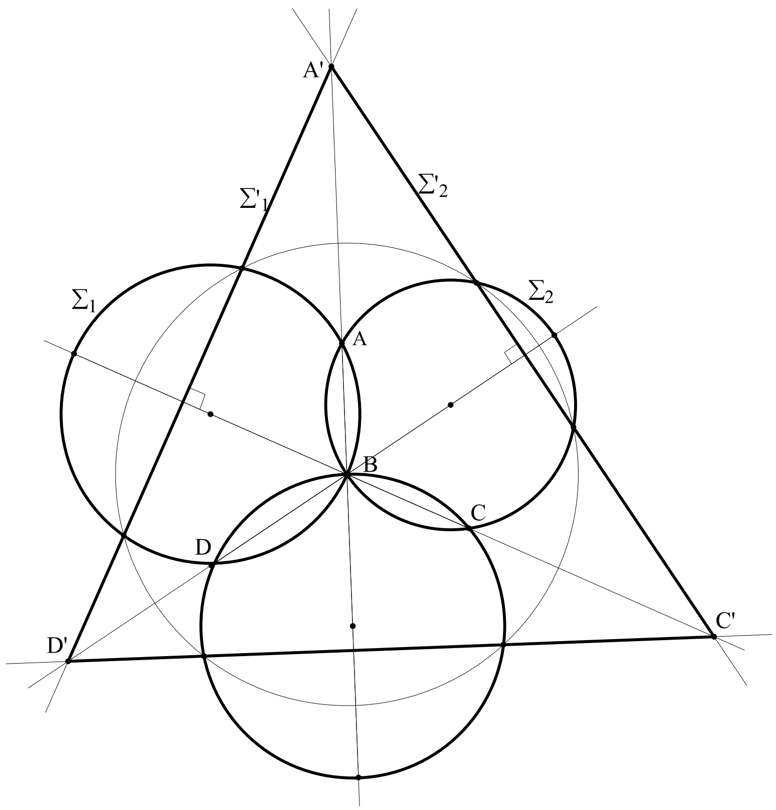


Figure 3.5.2

image AB of $A'B'$ must be orthogonal to the image circle BDC of $D'C'$. Hence AB is a diametral line of circle BDC and therefore passes through the center of circle BDC.

Our next example is a theorem due to Claudius Ptolemy, the great Alexandrian astronomer, who worked in the second century.

Theorem 3.5.1. *In a cyclic convex quadrilateral the product of the diagonals is equal to the sum of the products of the two pairs of opposite sides.*

There are several ways to prove Ptolemy's theorem. The ones I am familiar with require that a person can come up with a clever way to look at the problem. We will look at one of these before we show how simple inversion makes the proof. In the figure below, let $ABCD$ be the given cyclic quadrilateral with diagonals AC and BD.

Choose a point X on the diagonal BD so that angle BAX is equal to angle

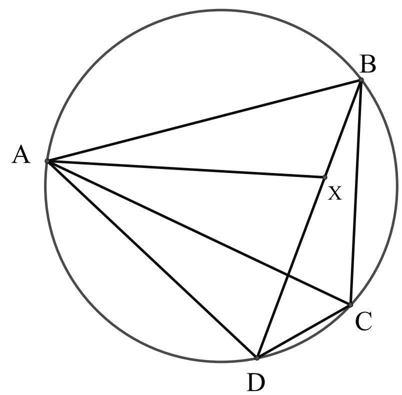


Figure 3.5.3

DAC. This is the clever part. Once this is done, the proof (which is left as an exercise) requires only knowledge of angle measure in circles and similar triangles.

On the other hand, if we know the properties of the inversion transformation, we know that the cyclic quadrilateral can be mapped from points on a circle to points on a line.

Proof. In Figure 3.5.4 $ABCD$ is a convex cyclic quadrilateral. We invert the quadrilateral and its circumcircle through an inversion $I(A, k)$, where k is greater than the diameter of the circumcircle of the quadrilateral. We let S represent the circle of inversion.

The vertices B, C, D map into the points B', C', D' lying on a straight line. Since B', C', D' are collinear, it follows that $B'D' = B'C' + C'D'$. Now by Theorem 3.4.13, we have that

$$\frac{r^2 BD}{AB \cdot AD} = \frac{r^2 BC}{AB \cdot AC} + \frac{r^2 CD}{AC \cdot AD}.$$

Multiplying the above equation by $\dfrac{AB \cdot AC \cdot AD}{r^2}$ gives

$$BD \cdot AC = BC \cdot AD + CD \cdot AB.$$

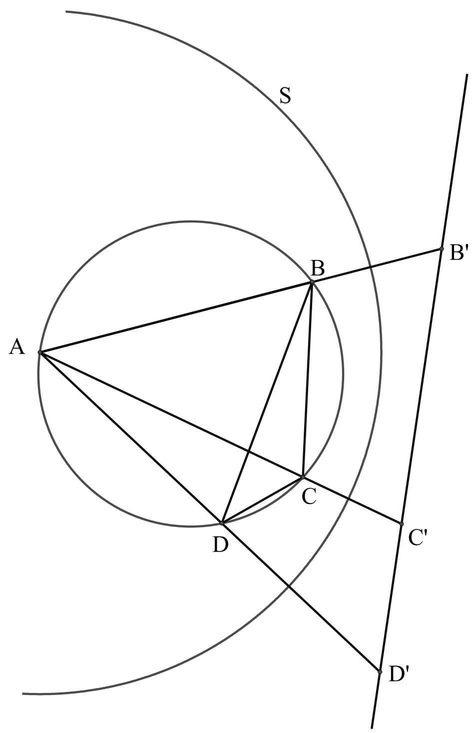


Figure 3.5.4

Note that for this inversion we could use any values for r. A convenient one would have been $r = 1$ so that $r^2 = 1$ in the above proof.

This next theorem is due to the Greek geometer Pappus (ca. 290-350) who lived and worked in Alexandria. He included this theorem in Book IV of his *Mathematical Collection* (ca. 300) and here he refers to this theorem as being already ancient.

Theorem 3.5.2. *Let X, Y, Z be three collinear points with Y between X and Z, and let C, C_1, K_0 denote semicircles, all lying on the same side of XZ, on XZ, XY, YZ as diameters. Let K_1, K_2, $K_3, \ldots$ denote circles touching C and C_1, with K_1 also touching K_0, K_2 also touching K_1, K_3 also touching K_2, and so on. Denote the radius of K_n by r_n, and the distance of the center of K_n from XZ by h_n. Then $h_n = 2nr_n$.*

Figure 3.5.5 illustrates the semicircles on XZ and the circles K_i.

Now if we let t_n represent the length of the tangent from X to the circle K_n, then we can invert the figure with the inversion $I(X, t_n^2)$. In Figure 3.5.6, we let

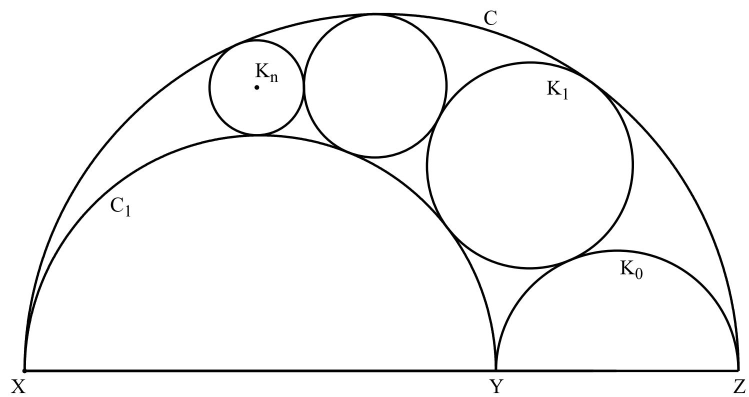


Figure 3.5.5

S represent the circle of inversion.

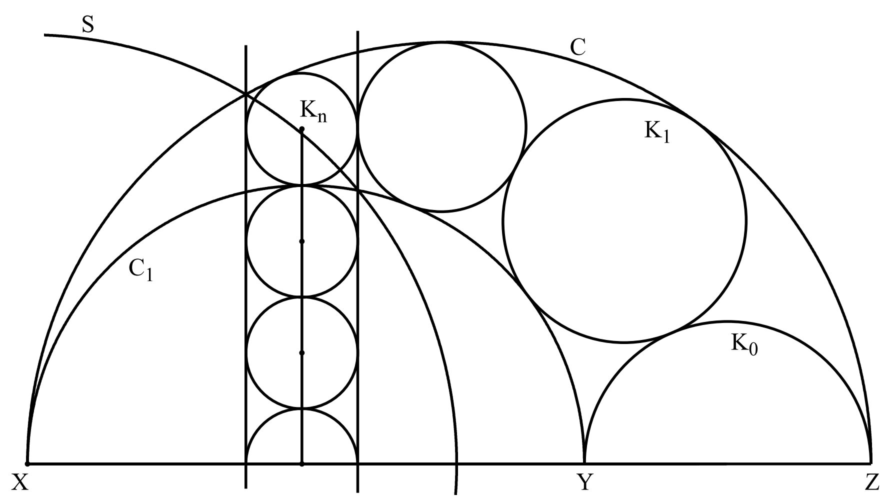


Figure 3.5.6

Since the tangent line to K_n from X is perpendicular to the radius of K_n, the two circles are orthogonal and, hence, under the inversion K_n maps onto itself. Furthermore, since C and C_1 are tangent to K_n, they will map into the two parallel vertical tangents perpendicular to XZ. Now the circles $K_1, K_2, \ldots, K_{n-1}$ must map into circles that are tangent to the two vertical lines and hence they must all have the same radius and they must also be tangent to the circles above and below them. Finally the semicircle K_0 must map into a semicircle. It is then clear that $h_n = 2nr_n$.

Theorem 3.5.3. *Two non-intersecting circles can always be mapped into two concentric circles.*

Proof. Let the given circles be C_1 and C_2. If C_1 and C_2 are already concentric, the proof is trivial. Hence we assume C_1 and C_2 are not concentric.

In Figure 3.5.7, C_1 and C_2 are the given circles. We begin by constructing the radical axis for C_1 and C_2. We then choose two points on the radical axis and construct circle D_1 and D_2 orthogonal to the two given circles C_1 and C_2. Since C_1 and C_2 are non-intersecting, the orthogonal circles will intersect the line of centers of C_1 and C_2 in two points P_1 and P_2. (Recall that all circles orthogonal to C_1 and C_2 will pass through the two points P_1 and P_2.) We now choose one of the points P_1 or P_2 as a center of inversion and invert circles C_1, C_2, D_1 and D_2. For purposes of illustration, P_1 is chosen as the center of a circle of inversion C. Through the circle C the four circles are inverted. Since D_1 and D_2 pass through the center of inversion, by Theorem 3.4.10, they will map into straight lines, D_1', D_2', whereas, by Theorem 3.4.11, circles C_1 and C_2 will map into circles, C_1', C_2'. Now, by Corollary 3.4.1(2), since D_1 and D_2 are orthogonal to C_1 and C_2, their images under inversion are orthogonal. In order for this to occur, the centers of both circles C_1 and C_2 must be the point of intersection of D_1' and D_2'. Recall that for a line to be orthogonal to a circle, the line must be a diagonal for the circle. Hence, C_1' and C_2' are concentric.

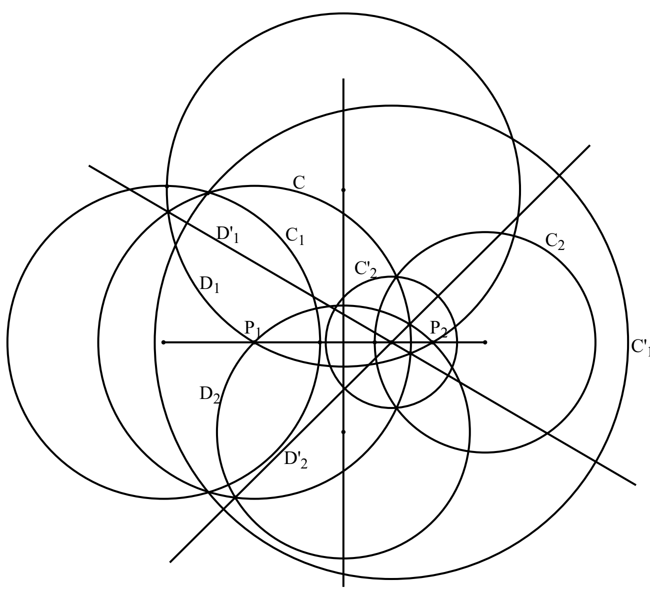


Figure 3.5.7: Theorem 3.5.3

Exercises

3.5.1. *Complete the first proof for Ptolemy's Theorem.*

3.5.2. *By construction, show that the inversion proof for Ptolemy's Theorem is still valid if r is chosen to be less than the diameter of the circumcircle of the quadrilateral.*

3.5.3. *Construct Figure 3.5.5. Explain the method of construction.*

3.5.4. *Construct Figure 3.5.6. Explain the method of construction.*

3.5.5. *Prove the Extension of Ptolemy's Theorem: In a convex quadrilateral $ABCD$*

$$BC \cdot AD + CD \cdot AB \geq BD \cdot AC,$$

with equality if and only if the quadrilateral is cyclic.

3.5.6. *Given two non-intersecting circles. Perform a construction that map them into concentric circles.*

Chapter 4

Other Geometries

4.1 Introduction

There are several geometries that fall under the classification of *non-Euclidean.* We will briefly examine three of them. The first two involve changing the parallel postulate in Euclidean geometry and the third is a coordinate geometry that involves changing the way distance in measured.

Euclid's fifth postulate, in essence, states that given a point P not on a line ℓ, there exists exactly one line through P parallel to ℓ. New consistent geometries can be obtained by replacing this postulate. We will briefly look at the history of challenging this postulate and the resulting geometry know as *hyperbolic geometry.* There are several models used to illustrate this geometry and we will look at one model that is closely related to our previous studies.

Because parallel lines exist in Euclidean geometry, it was assumed that they must exist in all geometries. However, in the mid nineteenth century Bernhard Riemann declared that one could reasonably assume that every pair of lines would intersect. This lead to the assumption of no parallel lines and resulted in the development of *elliptic geometry.* Here we will use the surface of a sphere to model elliptic geometry. We have to use artificial models for hyperbolic and elliptic geometry, since both of these geometries *look like* Euclidean geometry on a local level.

Our final example is a coordinate geometry called *Taxicab geometry.* In this geometry we change the definition of how we measure distance. Unlike the two previously mention geometries, this geometry can be studied at a very elementary level as well as at a very advanced level.

4.2 A Brief History of Hyperbolic Geometry

Since the history of the development of hyperbolic geometry is so intriguing, we will examine it briefly.

We begin with Euclid's fifth postulate:

> *If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely meet on that side on which are angles less than two right angles.*

The Greeks, Euclid included, were very skeptical of this postulate. It did not fit the mold of a postulate. The geometers who were disturbed by this postulate did not question its mathematical validity. They questioned only that it was not brief, simple, and self-evident, as postulates were supposed to be. It was felt by most that its complexity suggested that it should be a theorem instead of an assumption. This bothered Euclid as well. He postponed the use of the fifth postulate until Proposition 29 of Book I.

As a result the efforts prior to the nineteenth century were directed toward proving the fifth postulate as a theorem which depended on the remaining postulates. One of the earliest efforts to prove the parallel postulate was made by Proclus (410-485) in his *Commentaries on the First Book of Euclid's Elements.* In this book Proclus pointed out a fallacy contained in a proof by the astronomer Ptolemy. Proclus then gave his own proof in which he actually derived what we now call Playfair's axiom. The flaw in Proclus's reasoning was that he assumed that two parallel lines are everywhere the same distance apart. It can be shown that this assumption implies the parallel postulate. Thus Proclus was guilty of assuming what he was trying to prove.

In his book *Elements of Geometry*, John Playfair (1748-1819) stated an alternate form for the fifth postulate. This was not original with Playfair, but because of the popularity of his book, which went through ten editions between 1795 and 1846, the axiom bears his name. The popularity of the book probably influenced modern geometers to replace Euclid's fifth postulate with it.

> **Playfair's Axiom:** *Through a given point not on a given line, only one parallel can be drawn to the given line.*

It has been proved that Playfair's axiom is indeed equivalent to Euclid's fifth postulate.

Through the centuries, from the Greek period onward, many equivalent statements were discovered for the parallel postulate. Some were inadvertently assumed in trying to prove the fifth postulate and some were given in hopes of establishing an equivalent statement for the fifth postulate. Burton[1](pp 524-525)

[1]Burton, David M., *The History of Mathematics: An Introduction*, 5th ed., McGraw-Hill, 2003

lists some of these statements which have since been proven to be equivalent to the fifth postulate.

- A line that intersects one of two parallel lines intersects the other as well.
- There exist lines that are everywhere equidistant from one another.
- The sum of the angles of a triangle is equal to two right angles.
- For any triangle there exists a similar noncongruent triangle.
- Any two parallel lines have a common perpendicular.
- There exists a circle passing through any three noncollinear points.
- Two lines parallel to the same line are parallel to each other.

Many respected and honored mathematicians had to try their hand at proving the fifth postulate. In every instance these numerous and varied attempts failed. Invariably, proofs which seemed to succeed were found to be flawed. Like Proclus, many failed because their arguments were found flawed because of the use, open or hidden, conscious or unconscious of some assumption equivalent to the fifth postulate.

Some of the alleged proof, such as the one presented by John Wallis (1616-1703) in 1663 had a look of success. He proposed replacing Euclid's fifth postulate by the following:

> *To each triangle, there exists a similar triangle of arbitrary magnitude.*

This certainly fit the brief, simple, self-evident form that was desired for a postulate. Using this result and the other postulates of Euclid, Wallis was able to demonstrate that Playfair's axiom held, which was known to be the equivalent of the fifth postulate. Although Wallis' proof seemed perfectly reasonable, it rested on the assumption of the existence of two similar but noncongruent triangles. But this can be proved to be equivalent to the parallel postulate of Euclid. Thus, like others before him, he was assuming what he was to prove.

Girolamo Saccheri (1667-1733) made the first serious study of the logical consequence of an actual denial of the fifth postulate. His aim was to assume the postulate false and then derive it as a logical consequence. Saccheri began by considering a quadrilateral $ABCD$ (See Figure 4.2.1.) with sides AD and BC equal and perpendicular to the base AB. Saccheri then demonstrated that the summit angles at C and D were equal. From this he then declared three hypotheses: (1) $\angle C = \angle D > 90^\circ$ (obtuse case); (2) $\angle C = \angle D < 90^\circ$ (acute case); and (3) $\angle C = \angle D = 90^\circ$(right angle case). He then proceeded to prove that if one of these hypotheses were true for one of his quadrilaterals, then it would be true for every such quadrilateral. He then showed that if the obtuse case held then the sum of the angles of a triangle would be greater than two right angles. He had little difficulty in convincing himself that this was impossible.

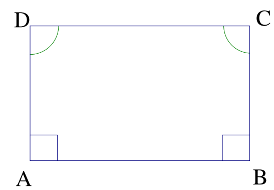


Figure 4.2.1: A Saccheri Quadrilateral

He next attacked the acute case. Here he had great difficulty in convincing himself of the fact that this case was impossible. In fact, he came very close to discovering the first non-Euclidean geometry; however, his faith in Euclid was so strong that he denied this possibility. He eventually declared that the acute hypothesis was false leaving only the right angle hypothesis which was equivalent to the fifth postulate. Saccheri's goal was to vindicate Euclid. However Burton (page 529) makes an interesting observation:-

> *Indeed had Saccheri actually accomplished his purpose and proved the parallel postulate from the remaining axioms of Euclidean geometry, he would not have vindicated Euclid. Quite to the contrary, he would have dealt a terrible blow to Euclid. Euclid was vindicated by the discovery of non-Euclidean geometry, for its existence demonstrated that the parallel postulate is independent of Euclid's other axioms, so that it truly widens the axiomatic base on which Euclid's geometry stands. We must admire the Great Geometer all the more; the introduction of the fifth postulate, so undecidedly unaxiomatic in appearance, yet an independent postulate, was a stroke of genius.*

Johann Lambert (1728 - 1777) was a German mathematician who tried to fix the arguments of Saccheri by proposing a quadrilateral in which three of the angles are right angles with the fourth angle being obtuse, acute or right.

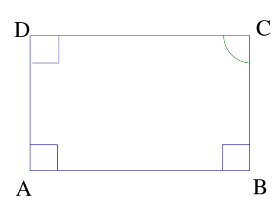


Figure 4.2.2: A Lambert Quadrilateral

Like Saccheri he was able to dispose of the obtuse case rather quickly; however, like Saccheri he had trouble with the acute case. But, unlike Saccheri, he realized he could not reach a contradiction for the acute case. In his investigation of this case he showed that in this new geometry the angle sum of a triangle increases when the area decreases. Lambert recorded his work in 1766 in an unpublished work titled *Theorie der Parallellinen* (Theory of Parallels). However he gave up his study of parallels when he realized that he could not successfully refute the hypothesis of the acute angle. Lambert is probably most famous for giving the first rigorous proof that π is irrational.

Another famous mathematician who tried to prove the fifth postulate was the French mathematician, Adrien-Marie Legendre (1752 - 1833). Legendre wrote *Eléments de géométrie* (Elements of Geometry) in 1795. This was a very popular book and went through 20 editions in France. It was also translated into English in 1819 by John Farrar, a professor of mathematics at Harvard, and went through ten American editions. It should be noted that in 1780 when the American Academy of Arts and Science was formed, John Adams made the suggestion (which was adopted by the Academy) that American educators follow those of France rather than England.

Legendre gave two theorems directed toward establishing the fifth postulate.

Theorem. *If the sum of the angles in a triangle is equal to two right angles, then Euclid's parallel postulate holds.*

Theorem. *The angle sum of a triangle is always less than or equal to* $180°$.

He then proved that the sum could not be less than $180°$. Therefore it must be equal to $180°$. Legendre concluded that because the sum of the angles of a triangle could neither be greater nor less than $180°$, it would have to be equal to $180°$. If this equality held, then the parallel postulate would follow. Unfortunately, there is a defect in Legendre's argument that the angle sum cannot be less than $180°$. To construct the sequence of triangles he used in the proof, he assumed that through any point in the interior of an angle it is always possible to draw a line that meets both sides of the angle. Although it is not immediately apparent, this assumption turns out to be equivalent to the parallel postulate!

By the 19th century mathematicians began to realize that, since great effort by great mathematicians failed to show the parallel postulate was dependent on the other postulates, it might be possible to replace the parallel postulate with another, which is contrary, and still develop a valid companion to Euclid's geometry.

When this idea finally hit, it came not to one but to three mathematicians. Carl Friedrich Gauss (1777 - 1855) in Germany, Nicolai Ivanovich Lobachevsky (1793 - 1856) in Russian and Jáos (John) Bolyai (1802 - 1860) in Hungary. All three seem to have based their work on that of Saccheri, who did not realize what he had since he was intent on proving the fifth postulate rather than challenging it.

Gauss appears to be the first to reach any substantial conclusions about a non-Euclidean geometry. However, Gauss was reluctant to publish his findings. Gauss, like Newton, disliked any kind of controversy so he was reluctant to let his discoveries be known. His feelings are best described in his own words in an 1824 letter to a colleague, Franz Taurunus:

> *The assumption that the sum of the three angles of a triangle is less than* 180° *leads to a curious geometry, quite different from our own [the Euclidean geometry], but thoroughly consistent, which I have developed to my satisfaction ...The theorems of this geometry appear to be paradoxical and, to the uninitiated, absurd; but calm, steady reflection reveals that they contain nothing impossible ...In any case, consider this a private communication, of which no public use or use leading to publicity is made. Perhaps I shall myself, if I have at some future time more leisure than in my present circumstances, make public my investigation.*[2]

Lobachevsky was the first of the three to publish a work on non-Euclidean geometry. His first publication was in 1829-1830 in the University of Kazan's monthly journal *Messenger* although he had orally communicated his ideas in 1826. It is worth noting here that prior to Lobachevsky's revelations in On the Foundations of Geometry, Johann Bartles, a former tutor and life-long friend of Gauss', was visiting Kazan and may well have had discussions with Lobachevsky about what Gauss had been doing with this new geometry of his. If such discussions took place, it should not detract from the work that Lobachevsky did.

In 1831 Bolyai, at his father's insistence published his work in an appendix to his father's work, *Tentamen*. It was first printed separately under the title *Appendix Scientiam Spatii Absolute Veram Exhibens* (Appendix Explaining the Absolute True Science of Space). The remarkable thing about this publication was that the body of the work contained only 24 pages. Again there is suspicion that Bolyai's ideas came from Gauss, since his father, Wolfgang Bolyai, was a friend of Gauss. Wolfgang Bolyai sent an advanced copy of his son's work to Gauss and receive the following reply from Gauss:-

> *If I begin by saying that I dare not praise this work, you will of course be surprised for a moment; but I cannot do otherwise. To praise it would amount to praising myself. For the entire content of the work, the approach which your son has taken, and the results to which he is led, coincide almost exactly with my own meditations which have occupied my mind for the past thirty or thirty-five years. ...It was my plan to put it all down on paper eventually, so that at least it would not perish with me. So I am greatly surprised to be spared the effort, and am overjoyed that it happens to be the son of my old friend who outstrips me in such a remarkable way.*[3]

Whatever the circumstances, these three remarkable mathematicians gave us a new and thoroughly consistent geometry. This new geometry bears the

[2]ibid. page 545
[3]ibid. pp 549-550

name Lobachevskian geometry, or hyperbolic geometry. Henri Poincaré(1854 - 1912) developed what is referred to as the Poincaré model for Lobachevskian plane geometry. With this model it can be shown that Lobachevskian plane geometry is consistent if Euclidean plane geometry is consistent. We choose to investigate this model because it uses geometric properties and constructions we have examined. In particular, the properties of orthogonal "circles."

4.3 The Poincaré Model

The purpose of this section is to introduce the Poincaré model. It is not intended to be a complete discussion of the model or of the consistency of Lobachevskian (hyperbolic) plane geometry. A set of axioms is said to be consistent if they and the resulting theorems, taken as a whole, produce no logical contradiction.

If we replace the parallel postulate (Postulate IV-1) in Hilbert's postulates by

> (IV-1)′ *Through a given point A not on a given line m there pass at least two lines that do not intersect line m.*

we will get a set of postulates for Lobachevskian plane geometry (also called hyperbolic geometry). To model this geometry, we will use the Poincaré model. In this model we use a circle Σ in the Euclidean plane to represent the Lobachevskian plane. A **point** in this plane will be any point in the interior of Σ. A **line** will be that part of the interior of Σ which lies on a "circle" (circle or straight line) orthogonal to Σ. In Figure 4.3.1 below, the circle Σ is a representation of the Lobachevskian plane. Points A, B, C are points on a line in Σ (part of a circle orthogonal to Σ) and point C is said to be between the points A and B. In the figure below, the entire circle representing the line containing points A and B is shown only for the purpose of showing the orthogonal circle. The only part of that circle that is really represented in the Poincaré model is the segment from S to T.

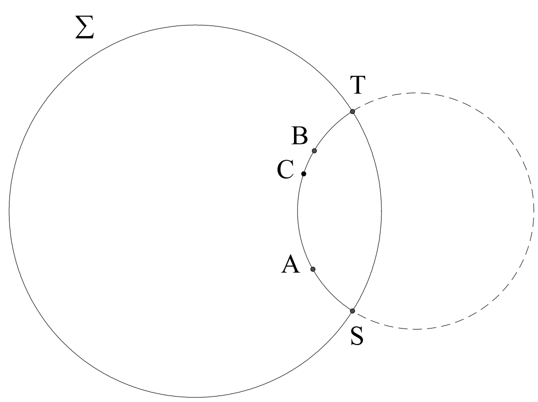


Figure 4.3.1: A line in the Poincaré Model

Definition. The **length of segment** $AB = \log(AB, TS) = \log\left[\frac{AT}{BT} \cdot \frac{BS}{AS}\right]$ where S and T are the points in which the "circle" containing the segment AB cuts Σ, S and T being labeled so that A is between S and B.

NOTE: (AB, TS) is the usual cross-ratio of the four points. The condition that S and T being labeled so that A is between S and B allows us to not have to deal with directed segments. It should also be noted that $(AB, TS) > 1$ and hence $\log(AB, TS) > 0$.

In Figure 4.3.2, m is a given line in Σ and P is any point in Σ not on m. There are two lines through P that do not intersect m. The line TPS' and the line $T'PS$. Recall that the points S, T, S' and T' are not in the interior of Σ and therefore cannot be points of intersection. They would be like ideal points for the extended Euclidean plane.

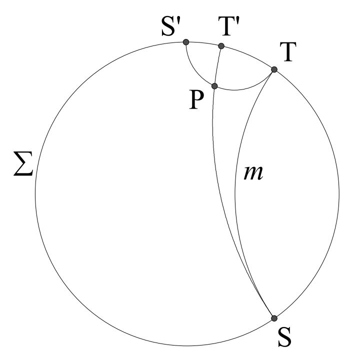


Figure 4.3.2: Parallel lines in the Poincaré Model

With this model we can actually construct examples of the Saccheri and Lambert quadrilaterals. In the Saccheri quadrilateral in Figure 4.3.3, the summit angles, $\angle C$ and $\angle D$, are acute. In the Lambert quadrilateral there are three right angles and one acute summit angle. Note that a line containing a diameter of Σ is a "circle" orthogonal to Σ. The complete line containing points A and B is drawn for the purpose of showing it is a diameter line for Σ.

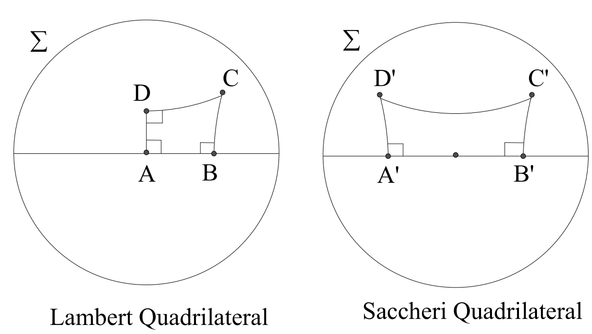


Figure 4.3.3: Quadrilaterals in the Poincaré Model

In Figure 4.3.4, we have a line in Σ containing the points A, B, C with C separating A and B. We will show that **length** $AB =$ **length** $AC+$ **length** CB, using the definition of the length of a segment. Now **length** $AB = \log(AB, TS) = \log\left[\frac{AT}{BT} \cdot \frac{BS}{AS}\right]$. If we multiply inside by something that is equal to 1, we will not alter the equality. Let us multiply by $\frac{CS}{CT} \cdot \frac{CT}{CS}$. Then

$$\begin{aligned}
\textbf{length } AB &= \log(AB, TS) \\
&= \log\left[\frac{AT}{BT} \cdot \frac{BS}{AS}\right] \\
&= \log\left[\frac{AT}{BT} \cdot \frac{BS}{AS} \cdot \frac{CS}{CT} \cdot \frac{CT}{CS}\right] \\
&= \log\left(\frac{AT}{CT} \cdot \frac{CS}{AS}\right)\left(\frac{CT}{BT} \cdot \frac{BS}{CS}\right) \\
&= \log\left(\frac{AT}{CT} \cdot \frac{CS}{AS}\right) + \log\left(\frac{CT}{BT} \cdot \frac{BS}{CS}\right) \\
&= \textbf{length } AC + \textbf{ length } CB.
\end{aligned}$$

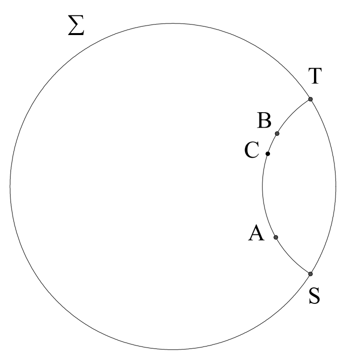


Figure 4.3.4: Distance in the Poincaré Model

Our final demonstration will be the construction of a triangle in the Poincaré model and to see how the angles are measured. First it should be noted that the measure of an angle of a triangle is the measure of the angle of intersection of the two "circles" forming a vertex for the triangle. This angle is measured in radians. In Figure 4.3.5, we have drawn three circles with centers P, Q and R which are orthogonal to Σ and meet in pairs in the three points A, B, C forming triangle ABC. All three circles are drawn in their entirety for clarity of construction. The angles are measured by constructing the tangents at the points of intersections (vertices) and measuring the angles of intersection. As drawn, $\angle ABC = 0.548$ radians (about $31.39°$), $\angle BAC = 0.718$ radians (about $41.13°$) and $\angle ACB = 1.520$ radians (about $87.12°$). The sum of the three angles is 2.786 radians (or $159.64°$).

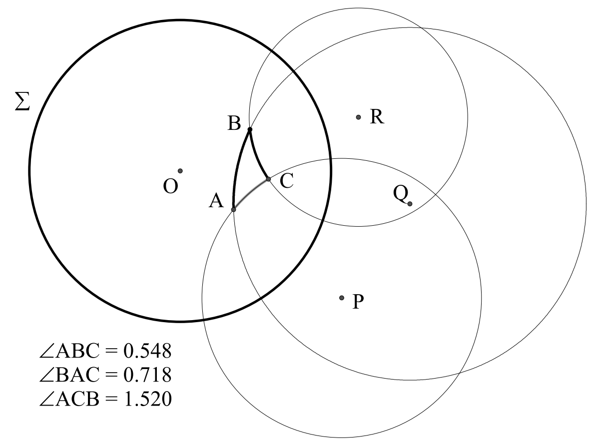


Figure 4.3.5: A Triangle in the Poincaré Model

Exercises

4.3.1. *Construct a triangle in the Poincaré Model. Explain how it is done.*

4.3.2. *Construct a rectangle in the Poincaré Model. Explain how it is done.*

4.3.3. *Construct a triangle and one of its altitudes in the Poincaré Model. Explain how it is done.*

4.3.4. *A circle in the Poincaré Model looks like a circle in the Euclidean plane except the centers do not coincide. Why is this so?*

4.3.5. *Explain, in terms of the Poincaré Model, why lines near the center of the Poincaré Model look straighter than lines closer to the boundary of the model.*

4.3.6. *Show that Euclid's first postulate holds in the Poicaré model:* Between any two distinct points a segment can be constructed.

Note.
There is a very interesting interactive Java applet for doing constructions in the Poicaré model. It can be found at the URL:

http://cs.unm.edu/∼joel/NonEuclid/NonEuclid.html

This applet can be run in any Java enabled browser. It is worth having for anyone who is interested in geometry - especially if one plans to teach.

4.4 Elliptic (Riemannian) Geometry

Merely for the purpose of completeness, we give a very brief discussion of elliptic geometry. Mainly, to show that there is a geometry with no parallel lines.

Hyperbolic geometry came about by assuming that there could be more than one parallel to a given line through a point not on the given line. Since Euclid allowed only one parallel and hyperbolic geometry allows more than one, the question then arises as to what happens if no parallels are allowed.

In 1854 Bernhard Riemann addressed this idea. His talk *Über die Hypothesen welche der Geometrie zu Grundee liegen* (On the Hypotheses that Underlie the Foundation of Geometry) is often cited as one of the highlights of modern mathematical history. In this talk he declared that one could reasonably assume that every pair of lines would meet at some finitely distant point. That is, the parallel postulate could be replaced by the following statement.

> Given a line ℓ and a point P not on ℓ, there exists no line parallel to ℓ passing through P.

A model for such a geometry could be the surface of a sphere with "lines" defined to be great circles on the surface of the sphere. Such lines are finite in length, since one of these lines would eventually return to is starting point. However, these lines are also unbounded since one could trace the line endlessly.

In Euclidean geometry the sum of the angles of a triangle sum to 180° and in hyperbolic geometry the sum is less than 180°. It is easily seen that in elliptic geometry, the sum is greater than 180°. For if we choose two points on the equator of the sphere to be two of the vertices of a triangle and the third vertex at the north pole, we clearly have a triangle whose angles sum to more than 180°.

It should be emphasized that *locally*, Euclidean, hyperbolic and elliptic geometry all look the same. We cannot tell if our chalkboard is in an Euclidean plane, hyperbolic plane or an elliptic plane. All triangles would appear to have angle measure of 180°. To get any variation from this, the triangle would have to be monstrously huge.

4.5 Taxicab Geometry

The geometry we refer to as *Taxicab geometry* is a coordinate geometry that was first considered by Hermann Minkowski (1864-1909) in the 19$^{\text{th}}$ century. Taxicab geometry can be of any dimension; however, we will restrict our attention to the Taxicab plane. For certain applications Taxicab geometry gives more realistic answers than Euclidean geometry. For example consider the problem facing Jane as she gets ready to walk to school. In the figure below, we see that if we let a city block represent a unit measure, then we see that the Euclidean distance from Jane's house to her school is $4\sqrt{2} \approx 5.66$ blocks. However, unless Jane has wings or a helicopter, she cannot follow the straight line path. Her actual distance from school is 8 blocks. We can see that there are many paths Jane can take from home to school which have a minimum distance of 8 blocks. The direct line from Jane's house to the school exists in both geometries; however, the difference is in the way we measure the length of the line.

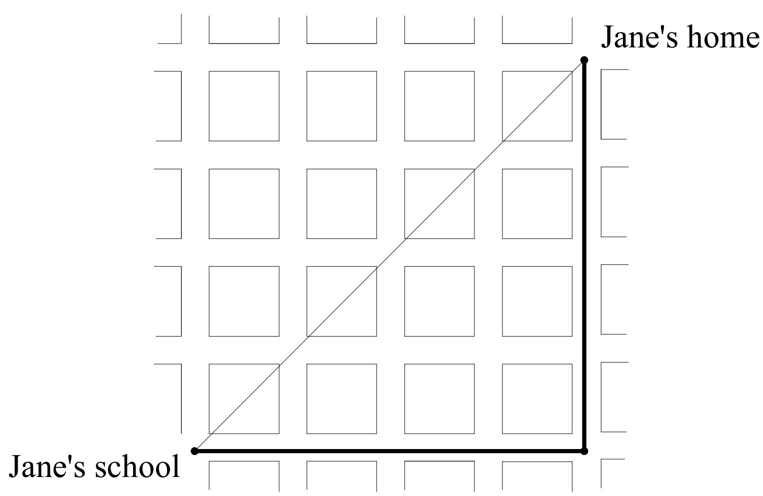


Figure 4.5.1: A Trip to School

In our everyday lives we use both Euclidean and Taxicab geometries. If we are interested in determining the length of a support to a building that is at a 45° angle to the building, we certainly need to use the Euclidean metric. On the other hand, if we are telling our new neighbor how far it is to Wal-Mart, we do it in terms of traveling city streets not as the crow flies.

We see that Taxicab geometry differs from coordinate Euclidean geometry in terms of the metric used. To determine the distance between two points in Taxicab geometry, we take the sum of the absolute differences of their coordinates. That is, in Taxicab plane, the distance between the points $P(x_0, y_0)$ and

$Q(x_1, y_1)$ is given by

$$d_T(P,Q) = |x_0 - x_1| + |y_0 - y_1|,$$

rather than the Euclidean metric

$$d_E(P,Q) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}.$$

In a sense, "Taxicab" geometry is a misnomer in that in this geometry we allow coordinates to be any real numbers. To be a true "Taxicab" geometry we would have to alter our definition of the metric to allow using the minimum *street distance* between points not at intersections.

It should be noted that Taxicab geometry satisfies all but one of Hilbert's axioms for Euclidean geometry. Lines in Taxicab geometry are the same as lines in Euclidean geometry. How their lengths are measured is what is different. Angles in Taxicab geometry are the same and measured the same as in Euclidean geometry. The Hilbert axiom that is not satisfied is the SAS congruence axiom.

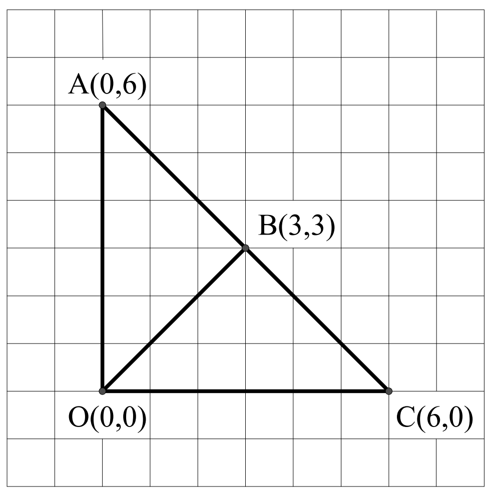


Figure 4.5.2: Side-Angle-Side Failure

Triangle AOC is a right isosceles triangle with two legs having length 6 and hypotenuse 12. Now triangle OBC is also a right isosceles triangle with legs of length 6 but the hypotenuse is of length 6 as well. These two triangle satisfy SAS but are clearly not congruent. We further note that triangle OBC is equilateral but not equiangular! It is also easy to see that SSS, which is not a Hilbert axiom, is not satisfied in Taxicab geometry.

From what we have discovered so far, it is clear that we must be careful not to try to project our visual notions of Euclidean geometry onto the Taxicab plane. Although we could use the same algebraic description for a circle with center O and radius r in both geometries, the resulting figure would not look the same. In both planes, a circle could be described as the set of all points P in the plane a distance r from the point O, which we would write as

$$\{P \mid d(O,P) = r\ \}.$$

Consider the circle in the Taxicab plane given by $\{P \mid d_T(O,P) = 4\}$. We cannot draw this with a compass, since the compass radius is using the Euclidean metric. What we get instead is the following

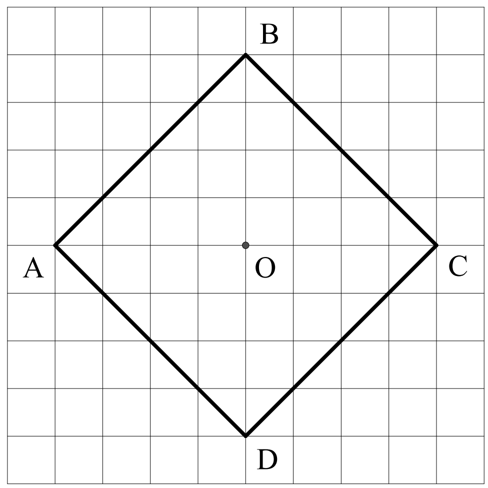


Figure 4.5.3: A Circle of Radius 4

In the above figure it is clear that $OA = OB = OC = OD = 4$. We can show that every point on the Taxicab circle is 4 units from O. We will show this is true for all points on the line joining A and B, as an example. If we place O at the origin, then A has coordinates $(-4,0)$ and B has coordinates $(0,4)$. The line segment AB has equation $y = x+4$ with $-4 \leq x \leq 0$. If $P(x,y)$ is any point on this line, then

$$d_T(O,P) = |x| + |x+4| = -x + x + 4 = 4.$$

The reader should verify the result for the other three sides.

Another shock is when we find the locus of points equidistant from two given points. There are two basic forms that we see. The first is not the line we would expect as the locus of points equidistant from the points $(-2,-2)$ and $(2,2)$. We see that there are two regions of points all of which are equidistant from the given points. See Figure 4.5.4

Suppose we let P be the point with coordinates $(-2,-2)$ and Q be the point $(2,2)$. Now suppose we wish to find all points $X(x,y)$ such that $d_T(P,X) = d_T(Q,X)$ with $x \leq -2$ and $y \geq 2$. We then have that $d_T(P,X) = |x+2|+|y+2|$ and $d_T(Q,X) = |x-2|+|y-2|$, so $-x-2+y+2 = -x+2+y-2$ which is an identity for $x \leq -2$ and $y \geq 2$. Hence all points (x,y) with $x \leq -2$ and $y \geq 2$ are equidistant from the points $(-2,-2)$ and $(2,2)$. The reader should verify that the remainder of the given points are equidistant from $(-2,-2)$ and $(2,2)$.

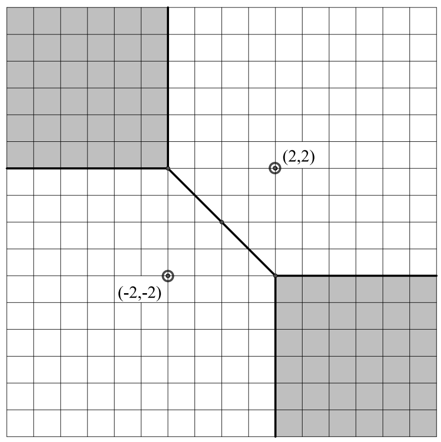


Figure 4.5.4: An Unexpected Locus

In the next figure, we see that the locus of points equidistant from $(0,0)$ and $(10,4)$ is a broken line. The reader should verify the validity of Figure 4.5.5.

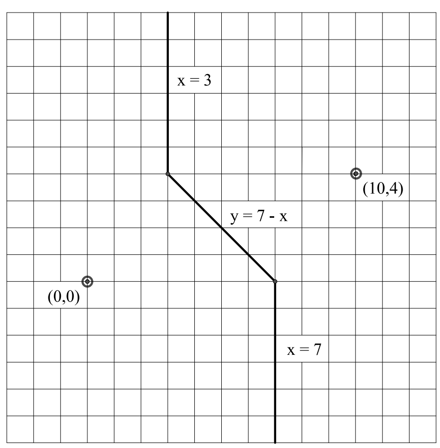


Figure 4.5.5: Another Unexpected Locus

In Euclidean geometry, we define an ellipse as the set of all points in the plane the sum of whose distance from two given points is constant. Using the same definition for Taxicab geometry, if A and B are the two fixed points and P any point on the ellipse, we would write

$$d_T(A, P) + d_T(B, P) = k.$$

If we let A be the point $(-3, 0)$, B be the point $(3, 3)$ and $k = 15$, we get the figure below. It looks like a squared off ellipse; however, notice where the foci are.

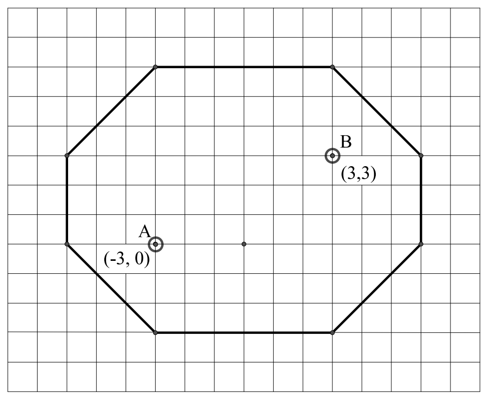


Figure 4.5.6: Ellipse: $d_T(A, P) + d_T(B, P) = 15$

On the other hand, if we put both foci on the $x-$axis, we get a more familiar figure.

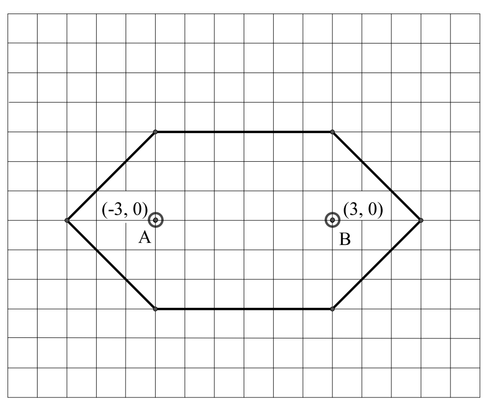


Figure 4.5.7: Ellipse: $d_T(A, P) + d_T(B, P) = 9$

Example 4.5.1. *Graph the ellipse* $d_T(AP) + d_T(BP) = 13$ *with* $A(-3, 2)$ *and* $B(4, 2)$.

Solution. We begin by partitioning the Taxicab plane into six regions using the points A and B as the points for partitioning.

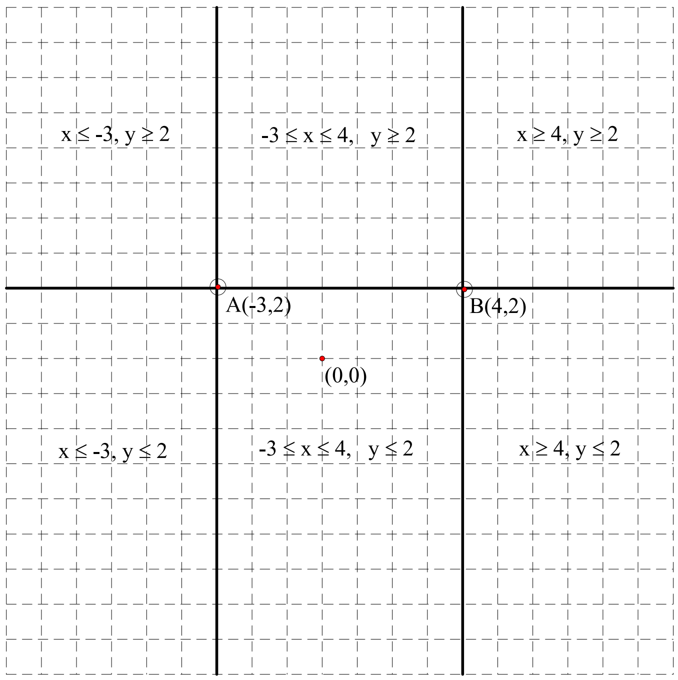


Figure 4.5.8: Partitioned Plane

The equation of the ellipse becomes

$$d_T(AP)+d_T(BP) = |x+3|+|y-2|+|x-4|+|y-2| = |x+3|+|x-4|+2|y-2| = 13.$$

We must now evaluate the above equation for $P(x, y)$ in each of the six regions.

For the region given by $-3 \leq x \leq 4, \quad y \leq 2$, we have

$$\begin{aligned} |x+3| + |x-4| + 2|y-2| &= (x+3) + (-x+4) + 2(-y+2) \\ &= x + 3 - x + 4 - 2y + 4 \\ &= -2y + 11 \end{aligned}$$

From $-2y + 11 = 13$ we get $y = -1$. This region therefore contains the line $y = -1$ for $-3 \leq x \leq 5$.

For the region given by $x \geq 4, \quad y \leq 2$, we have

$$\begin{aligned}|x+3|+|x-4|+2|y-2| &= (x+3)+(x-4)+2(-y+2)\\ &= 2x-1-2y+4\\ &= 2(x-y)+3\end{aligned}$$

From $2(x-y)+3=13$ we get $x-y=5$. This region therefore contains the line $y=x-5$ for $x \geq 4, \quad y \leq 2$.

For the region given by $x \geq 4, \quad y \geq 2$, we have

$$\begin{aligned}|x+3|+|x-4|+2|y-2| &= (x+3)+(x-4)+2(y-2)\\ &= 2x-1+2y-4\\ &= 2(x+y)-5\end{aligned}$$

From $2(x+y)-5=13$ we get $x+y=9$. This region therefore contains the line $y=-x+9$ for $x \geq 4, \quad y \geq 2$.

For the region given by $-3 \leq x \leq 4, \quad y \geq 2$, we have

$$\begin{aligned}|x+3|+|x-4|+2|y-2| &= (x+3)-(x-4)+2(y-2)\\ &= 2y+3\end{aligned}$$

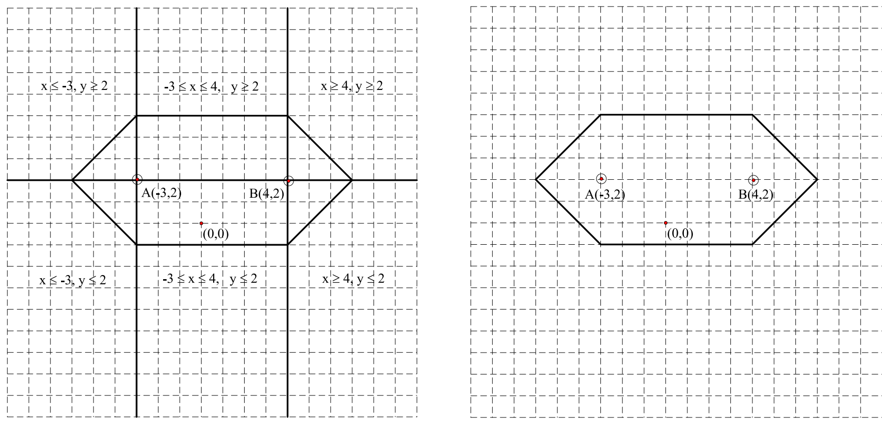
From $2y+3=13$, we get the line $y=5$ for $-3 \leq x \leq 4$

Using similar arguments for the remaining regions, we obtain the following.

For the region given by $x \leq -3, \quad y \geq 2$, we have the line $y=x+8$ for $x \leq -3, \quad y \geq 2$.

For the region given by $x \leq -3, \quad y \leq 2$, we have the line $y=-x-4$ for $x \leq -3, \quad y \leq 2$.

The graph of this ellipse appears below. First with the six regions identified and then by itself.



In Euclidean geometry, we define a hyperbola as the set of all points in the plane the difference of whose distance from two given points is constant. Using a minor variation of the same definition for Taxicab geometry, if A and B are the two fixed points and P any point on the hyperbola, we would write

$$|d_T(A,P) - d_T(B,P)| = k.$$

Here we need to use the absolute value to obtain all possibilities. The hyperbola in the Taxicab plane can take on a variety of shapes depending on the choice of foci and k.

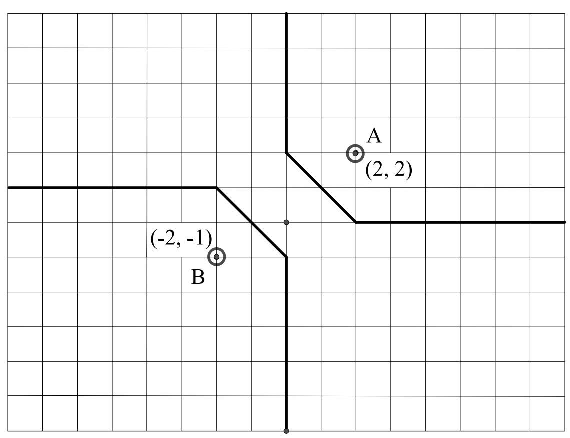


Figure 4.5.9: Hyperbola: $\mid d_T(A,P) - d_T(B,P) \mid = 3$

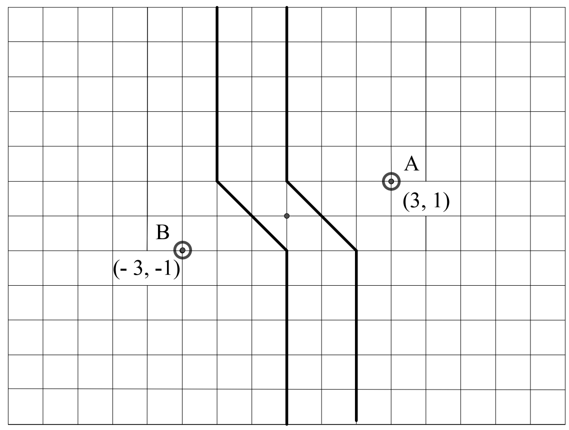


Figure 4.5.10: Hyperbola: $|d_T(A, P) - d_T(B, P)| = 2$

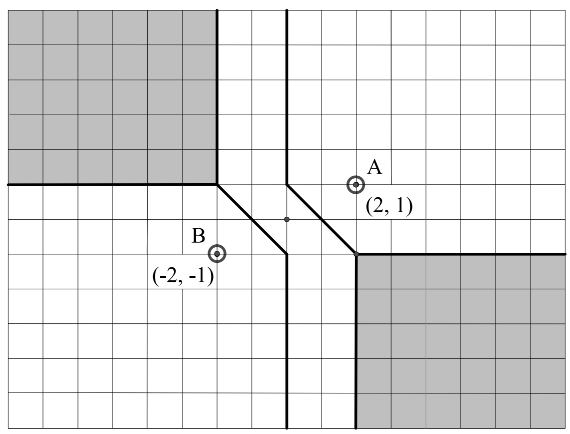


Figure 4.5.11: Hyperbola: $|d_T(A, P) - d_T(B, P)| = 2$

In the above examples, we see that the choice of the foci creates very different looking figure even when the same value of k is used. These are only a few of the possibilities.

Example 4.5.2. *Graph the hyperbola* $|d_T(AP) - d_T(BP)| = 6$ *with* $A(-3, -1)$ *and* $B(5, 3)$.

Solution. We begin by partitioning the Taxicab plane into nine regions using the points A and B as opposite corners of a rectangle. We then apply the x and y values in each region to the definition of our hyperbola.

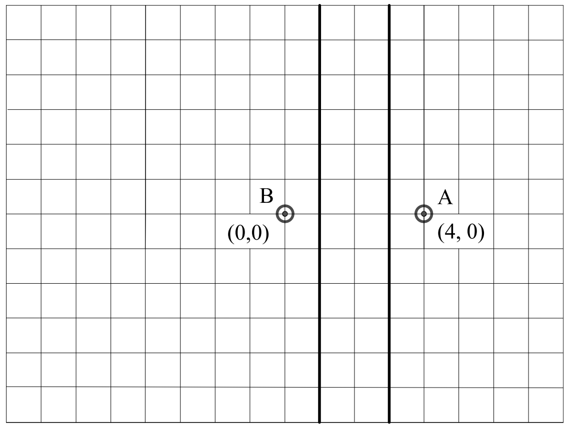


Figure 4.5.12: Hyperbola: $|d_T(A, P) - d_T(B, P)| = 2$

The equation of the hyperbola becomes

$$|d_T(AP) - d_T(BP)| = \Big||x+3| + |y+1| - |x-5| - |y-3|\Big| = 6.$$

We must now evaluate the above equation for $P(x, y)$ in each of the nine regions. For the region given by $-3 \leq x \leq 5, \quad -1 \leq y \leq 3$, we have

$$\begin{aligned}\Big||x+3| + |y+1| - |x-5| - |y-3|\Big| &= |(x+3) + (y+1) - (-x+5) - (-y+3)| \\ &= |x+3+y+1+x-5+y-3| \\ &= |2x+2y-4| \\ &= 2|x+y-2|\end{aligned}$$

From $2|x+y-2| = 6$ we get $x+y-2 = \pm 3$ and this yields $x+y = 5$ or $x+y = -1$. This region therefore contains the lines $y = -x+5$ and $y = -x-1$ for $-3 \leq x \leq 5, \quad -1 \leq y \leq 3$.

For the region given by $x \geq 5, \quad -1 \leq y \leq 3$, we have

$$\begin{aligned}\Big||x+3| + |y+1| - |x-5| - |y-3|\Big| &= |(x+3) + (y+1) - (x-5) - (-y+3)| \\ &= |x+3+y+1-x+5+y-3| \\ &= |2y+6| \\ &= 2|y+3|\end{aligned}$$

From $2|y+3| = 6$, we get $|y+3| = 3$ and $y = 0$ or $y = -6$; however, $y = -6$ is out of the range $-1 \leq y \leq 3$ and is rejected. This region therefore contains the

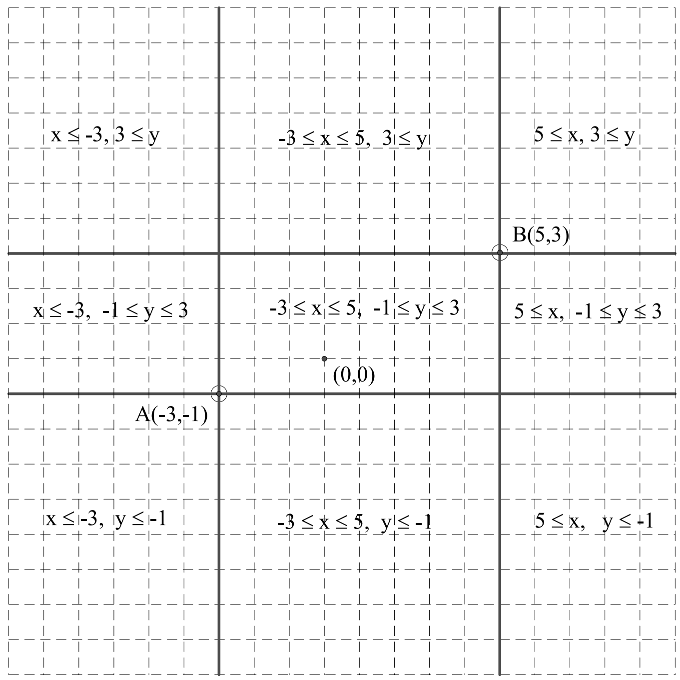


Figure 4.5.13: Partitioned Plane

line $y = 0$ for $x \geq 5$.

For the region given by $x \geq 5, \quad y \geq 3$, we have

$$\begin{aligned}\Big||x+3| + |y+1| - |x-5| - |y-3|\Big| &= |(x+3) + (y+1) - (x-5) - (y-3)| \\ &= |x+3+y+1-x+5-y+3| \\ &= |12|\end{aligned}$$

Since $|12| \neq 6$, this region *contains no points on the hyperbola.*

For the region given by $-3 \leq x \leq 5, \quad y \geq 3$, we have

$$\begin{aligned}\Big||x+3| + |y+1| - |x-5| - |y-3|\Big| &= |(x+3) + (y+1) - (-x+5) - (y-3)| \\ &= |x+3+y+1+x-5-y+3| \\ &= |2x+2| \\ &= 2|x+1|\end{aligned}$$

From $2|x+1| = 6$, we get $|x+1| = 3$ and this gives $x = 2$ or $x = -4$; however, $x = -4$ is outside the range $-2 \leq x \leq 5$, so we have the line $x = 2$ for $y \geq 3$ for this region.

Using similar arguments for the remaining regions, we obtain the following.

For region $x \leq -3, \quad y \geq 3$ we find no points in this region satisfying the equation of the hyperbola.

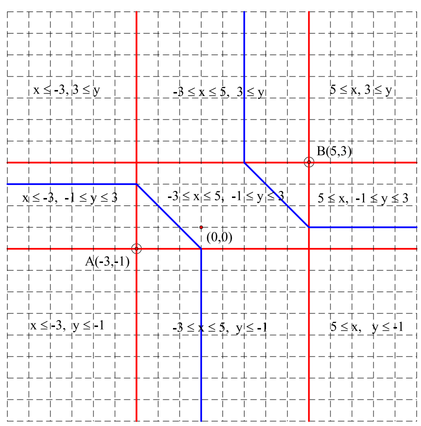
For the region $x \leq -3, \quad -1 \leq y \leq 3$, we find the line $y = 2$ for $x \leq -3$

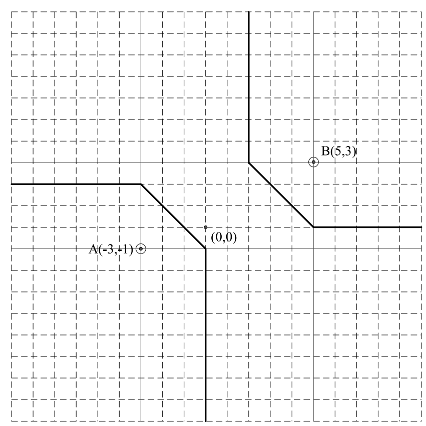
For the region $x \leq -3, \quad y \leq -1$ we again find no points in this region satisfying the equation of the hyperbola.

For the region given by $-3 \leq x \leq 5, \quad y \leq -1$, we find the line $x = 0$ for $y \leq -1$

For the region given by $x \geq 5, \quad y \leq -1$ we again find no points in this region satisfying the equation of the hyperbola.

The graph of this hyperbola appears below. First with the nine regions identified and then by itself.





Exercises

4.5.1. *Construct the set of points equidistant from the points* $(-2, -3)$ *and* $(1, 0)$.

4.5.2. *Construct the set of points equidistant from the points* $(1, 1)$ *and* $(3, 3)$.

4.5.3. *In Figure 4.5.3, show that all points on the line joining* B *and* C *are* 4 *units from* O.

4.5.4. *Construct the Taxi-circle with center* $(-2, 3)$ *and radius* 5.

4.5.5. *In Figure 4.5.4, show that all points in the lower, shaded area are equidistant from the two given points.*

4.5.6. *Verify the validity of the result in Figure 4.5.5.*

4.5.7. *Given the points* $(-1, 7)$, $(-3, -1)$, $(3, -3)$. *Find the point equidistant from all three points.*

4.5.8. *Construct the Taxicab circumcircle for the triangle with vertices* $(-1, 7)$, $(-3, -1)$, *and* $(3, -3)$.

4.5.9. *In the figure below, the grid is partitioned so that all the points in the region about A are closest to A, all the points in the region about B are closest to B and all the [points in the region about C are closest to C. Explain how these regions are constructed.*

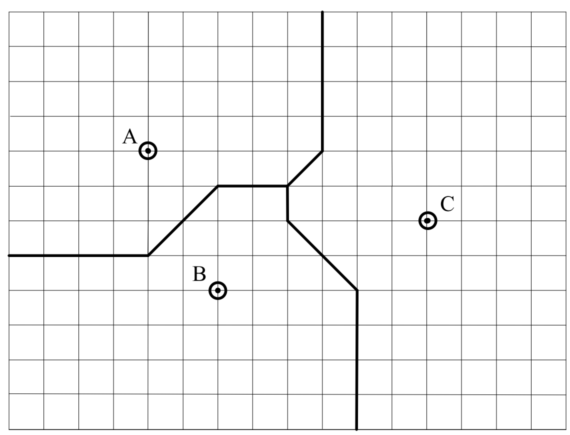


Figure 4.5.14: A Voronoi Diagram for Three Points

4.5.10. *The regions in Figure 4.5.14 compose what is called a Voronoi diagram. These diagrams were named after George Voronoi (1868 - 1908) even though they had been considered as early as the 17*th *Century. Construct a Voronoi diagram for closest points using the points* $A(-2, -3)$, $B(0, 5)$ *and* $C(2, -1)$. *What point is equidistant from the three given points?*

4.5.11. *A Taxicab parabola is the set of all points equidistant from a given point F and a given line* ℓ*; that is, the set of all points P for which* $d_T(P, F) = d_T(P, \ell)$. *Construct the parabola using the point* $F(1, 2)$ *and the line* $y = -2$.

4.5.12. *Construct an ellipse satisfying* $d_T(a, P) + d_T(B, P) = 10$, *with A being the point* $(2, 4)$ *and B the point* $(6, 6)$.

4.5.13. *Construct the hyperbola satisfying* $|d_T(A, P) - d_T(B, P)| = 2$, *with A being the point* $(-2, -1)$ *and B the point* $(2, 1)$.

4.5.14. *Construct the hyperbola satisfying* $|d_T(A,P) - d_T(B,P)| = 3$*, with* A *being the point* $(1,1)$ *and* B *the point* $(5,4)$.

Appendices

Appendix A
Some Basic Constructions

Duplicating an Angle

In Figure 1, the object is to construct an angle at P equal to the angle at A.

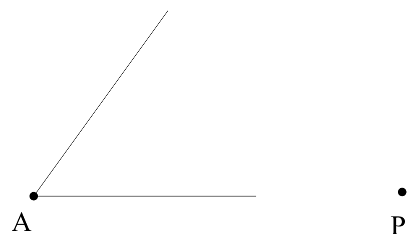


Figure 1.

First we draw a line through P for the initial side of the angle. At A strike an arc cutting the sides of the angle A in points B and C. Without changing the radius of your compasss, strike an arc of the same radius at P cutting the line through P at Q.

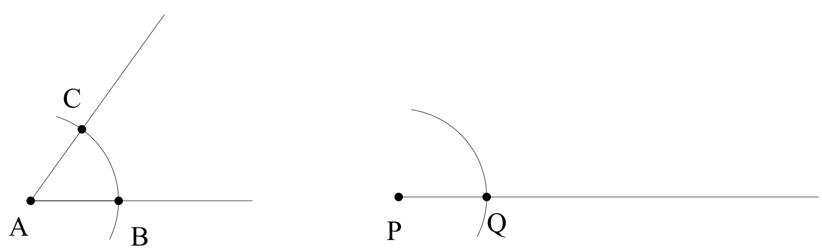


Figure 2.

With compass at point B, measure the distance BC. With this radius, mark off QR. Draw a line through PR. Angle QPR is the desired angle.

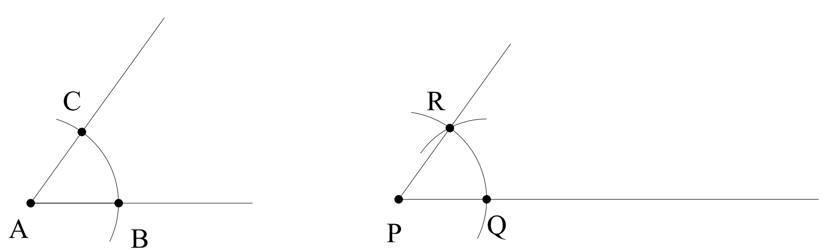


Figure 3.

Constructing a Perpendicular to a Given Line

We first construct a perpendicular from a point P not one the line to the line ℓ.

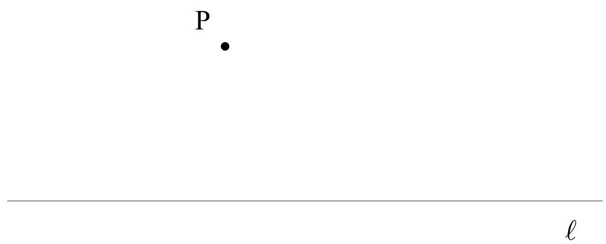


Figure 4.

We set our compass with a radius that exceeds the distance from P to line ℓ and strike an arc cutting the line ℓ in the points A and B.

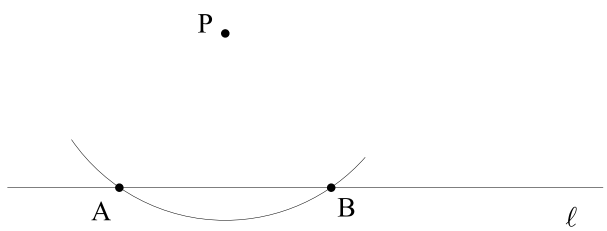


Figure 5.

With the same radius, strike arcs from A and B meeting in the point Q. The line through P and Q is the desired perpendicular.

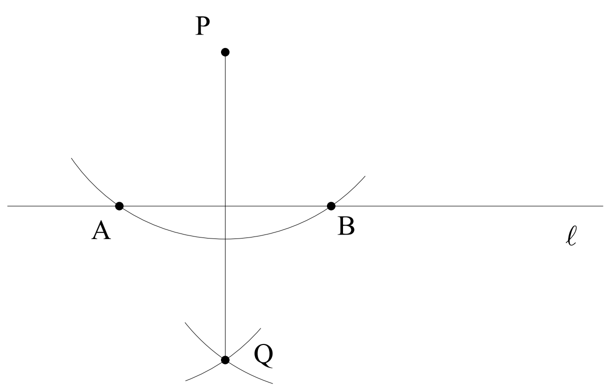


Figure 6.

NOTE. The radius used when striking the arcs from A and B need not be equal to PA. All that is required is that this radius be the same for both points A and B and be greater than $\frac{1}{2}AB$.

We now examine the case where the point P is on the line ℓ.

Figure 7.

In this case, we choose any convenient radius and strike arcs from P intersecting the line ℓ in the points A and B.

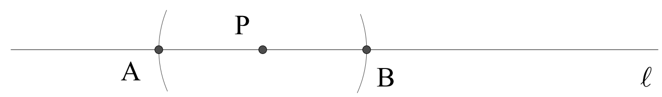


Figure 8.

At A and B we strike equal arcs whose measure is bigger than PA to meet in a point Q. The line PQ is the desired perpendicular.

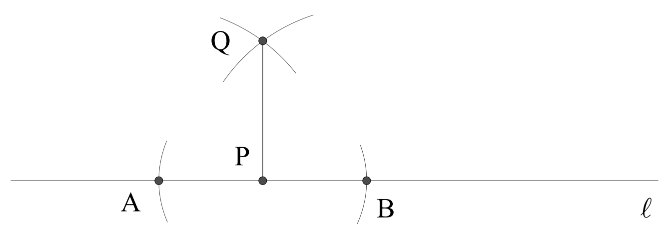


Figure 9.

Constructing a Parallel to a Given Line

We wish to construct a line through a given point P to a given line ℓ.

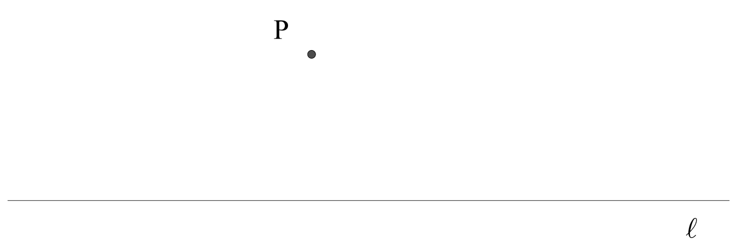


Figure 10.

We first review a little of our knowledge of parallel lines. We note that is a transversal t cuts a pairs of parallel lines ℓ and m, the corresponding angles are equal, as seen in Figure 11.

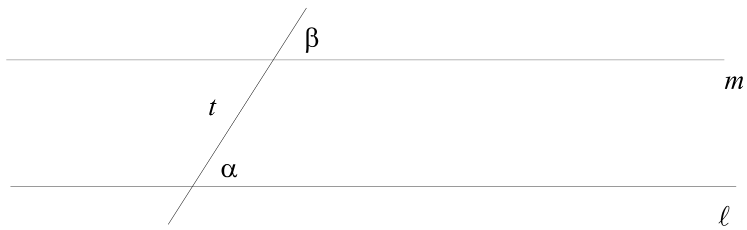


Figure 11.

If we draw a line through P cutting line ℓ in a point Q, we see that all we need do is duplicate the angle at Q at the point P to get the desired parallel line.

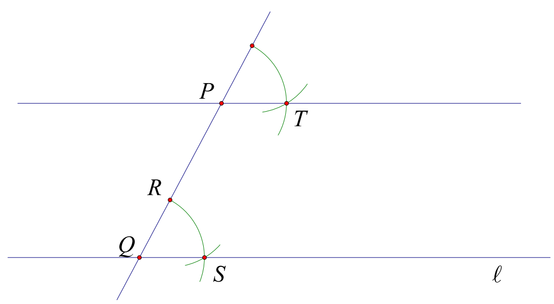


Figure 12.

NOTE. Since this construction works for any trnsversal, some prefer to drop a perpendicular from P to ℓ and then construct the perpendicular at P to that transversal.

Constructing an Angle Bisector

To bisect a given angle A

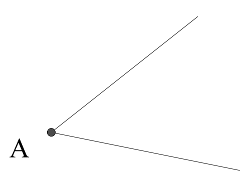


Figure 13.

Set compass point at A and strike an arc cutting the sides of the angle in the points B and C.

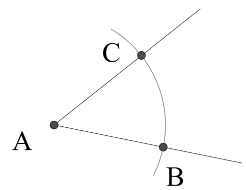


Figure 14.

At points B and C strike arcs of equal radii meeting in a point D. The line AD is the bisector of the angle A.

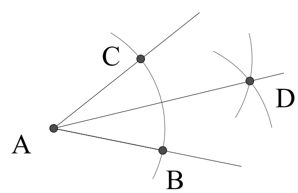


Figure 15.

Appendix B

Concurrence of the Medians of a Triangle

We will prove a well-known theorem in geometry by using the old methods of Euclidean geometry and then you can compare this proof with a proof using analytic geometry. (Of course, Ceva's Theorem, which came many centuries after Euclid, gives the easiest proof of all.) The theorem in question is the median concurrence theorem. That is, the medians of a triangle meet in a point.

Euclid's Method

For the Euclidean proof we need the following theorems (listed in the order they are usually presented in an elementary geometry course). We state these theorems without proof and realize that these theorems, in turn, depend on other theorems as well.

Theorem 1. *In a plane, two lines parallel to a third are parallel to each other.*

Theorem 2. *The diagonals of a parallelogram bisect each other.*

Theorem 3. *If two nonconsecutive sides of a quadrilateral are both congruent and parallel, then the quadrilateral is a parallelogram.*

Theorem 4. *The segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long as the third side.*

Now for the target theorem.

Theorem 5. *The medians of a triangle are concurrent and the point of concurrence is two-thirds of the distance from any vertex to the midpoint of the opposite side.*

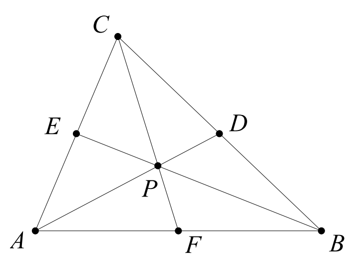


Figure B-1.

We are given triangle ABC with medians $AD,\ BE,$ and CF.
We wish to prove that $AD,\ BE,$ and CF are concurrent at P and that

$$AP = \frac{2}{3}AD;\ BP = \frac{2}{3}BE;\ \text{and}\ CP = \frac{2}{3}CF.$$

Proof. In Figure 2, let AD and BE intersect at P and let M and N be the midpoints of AP and BP, respectively. Draw DE, EM, MN, ND. Since DE is the line joining the midpoints of sides BC and CA of triangle ABC, we have, by Theorem 4, that $DE \parallel AB$ and $DE = \frac{1}{2}AB$. Since MN is the line joining the midpoints of sides PA and PB of triangle APB, we have, by Theorem 4, that $MN \parallel AB$ and $MN = \frac{1}{2}AB$.

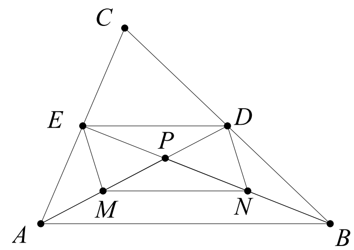


Figure B-2.

Furthermore, by Theorem 1, $DE \parallel NM$ since they are both parallel to AB. It now follows from Theorem 3 that $DEMN$ is a parallelogram. Hence, by Theorem 2, the diagonals DM and EN bisect each other. That is, $DP = PM$ and $EP = NP$, so

$$AM = MP = PD \text{ and } BN = NP = PE.$$

Therefore,

$$AP = \frac{2}{3}AD \text{ and } BP = \frac{2}{3}BE.$$

Next we let AD and CF intersect at P' (see Figure 3). By the same argument, using points R and S and parallelogram $DFSR$, we can establish that

$$CP' = \frac{2}{3}CF \text{ and } AP' = \frac{2}{3}AD.$$

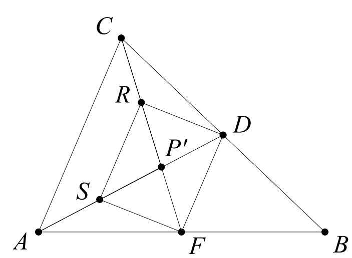


Figure B-3.

We now have that $AP = \frac{2}{3}AD$ and $AP' = \frac{2}{3}AD$. The first says that P is a point on AD such that P is two-thirds the distance to A from D and the second says that P' is a point on AD such that P' is two-thirds of the distance from A to D. Hence P and P' must be the same point. Thus P lies on all three medians; that is, the medians are concurrent at P. Furthermore, $AP = \frac{2}{3}AD$; $BP = \frac{2}{3}BE$; and $CP = \frac{2}{3}CF$.

The proof is not long, but it does require a good understanding of several theorems of Euclidean geometry.

Analytic Geometry Method

We now turn our attention to proving the same theorem using the tools of analytic geometry. The tools we need here are: knowledge of the Cartesian plane; the midpoint formula for a line segment; the point-slope formula for an equation of a line; solving two equations in two unknowns; and the distance formula.

Theorem 6. *The medians of a triangle are concurrent and the point of concurrence is two-thirds of the distance from any vertex to the midpoint of the opposite side.*

Proof. Without loss of generality, we may align our triangle so that one vertex is at the origin and one side lies on the x-axis. We give the vertices coordinates $A(0,0)$, $B(a,0)$ and $C(b,c)$. We next find the midpoints of sides $AB(\frac{a}{2},0)$, $BC(\frac{a+b}{2},\frac{c}{2})$ and $CA(\frac{b}{2},\frac{c}{2})$.

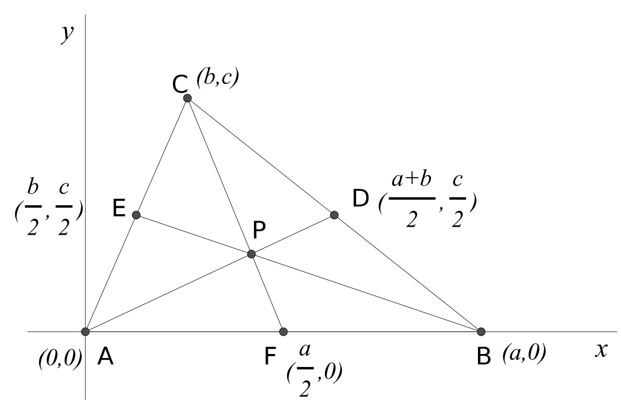


Figure B-4.

We next find the equations for the lines containing the segment AD and the segment BE. The slope of AD is $m_{AD} = \frac{c}{a+b}$ and the equation of the line is $y = \frac{c}{a+b}x$. The slope of BE is $m_{BE} = \frac{c}{b-2a}$ and the equation of the line is $y = \frac{c}{b-2a}(x-a)$.

Solving these two equations will give us the coordinates of the point P. Setting the y-values equal, we get, upon solving for x, $x = \frac{a+b}{3}$. From the equation for line AD, if $x = \frac{a+b}{3}$ then $y = \frac{c}{3}$. Therefore the coordinates of P are $(\frac{a+b}{3}, \frac{c}{3})$.

Now we need to find an equation for the line containing the segment CF and show that $P(\frac{a+b}{3}, \frac{c}{3})$ is on that line. The slope of CF is $m_{CF} = \frac{2c}{2b-a}$ and the equation of the line is $y = \frac{2c}{2b-a}(x - \frac{a}{2})$. Letting $x = \frac{a+b}{3}$ we get

$$y = \frac{2c}{2b-a}\left(\frac{a+b}{3} - \frac{a}{2}\right) = \frac{c}{3}$$

and P is on the line containing the segment CF. Therefore the medians are concurrent.
We can show that the point of concurrence, P, is two-thirds of the distance from any vertex to the midpoint of the opposite side by using the distance formula.

$$\begin{aligned} |AP| &= \sqrt{\frac{(a+b)^2}{9} + \frac{c^2}{9}} = \frac{\sqrt{a^2+b^2+c^2+2ab}}{3} \\ |AD| &= \sqrt{\frac{(a+b)^2}{4} + \frac{c^2}{4}} = \frac{\sqrt{a^2+b^2+c^2+2ab}}{2} \end{aligned}$$

and we see that $|AP| = \frac{2}{3}|AD|$. The others may be shown in a similar manner.

The method of analytic geometry is very straight forward using the basic tools of algebra. People seem to have a much better grasp of the tools of algebra than they do of the theorems of Euclidean geometry. Thus, when Descartes gave the mathematical world analytic geometry, he freed the mathematician from the rigors of Euclidean geometry.

Appendix C

Illustrations for Section 2.8

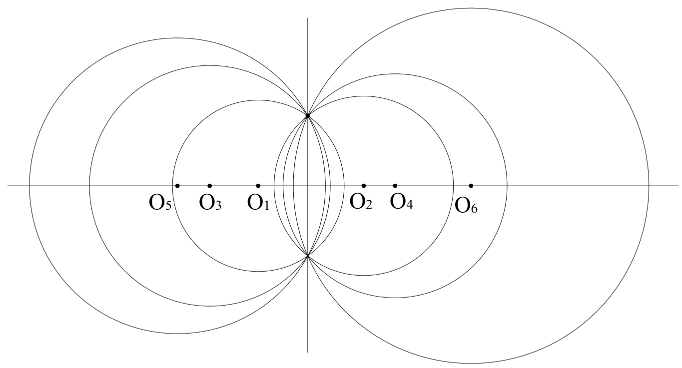


Figure C-1: An Intersecting Coaxial Pencil

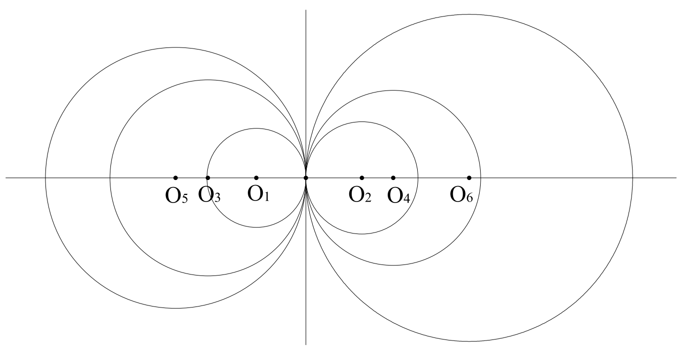


Figure C-2: A Tangent Coaxial Pencil

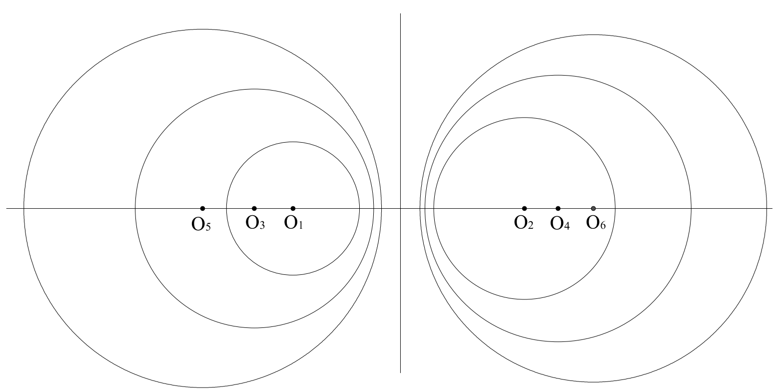


Figure C-3: A Non-intersecting Coaxial Pencil

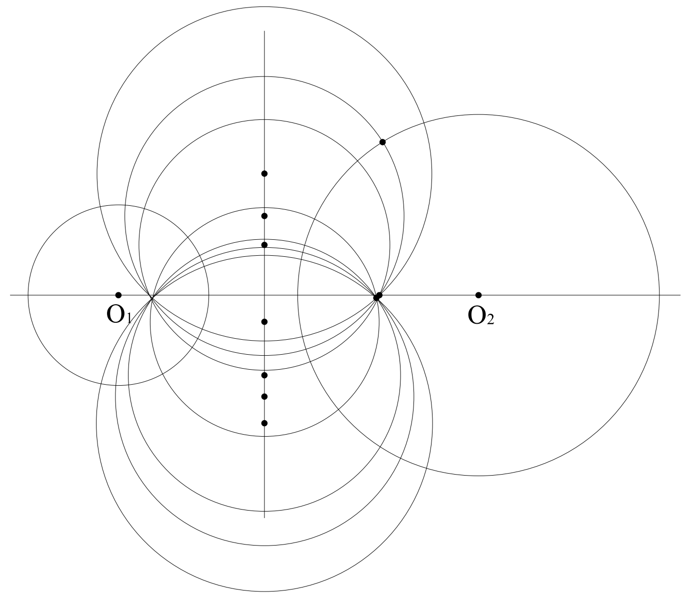


Figure C-4: An Intersecting Coaxial Pencil Orthogonal to Two Given Non-intersecting Circles

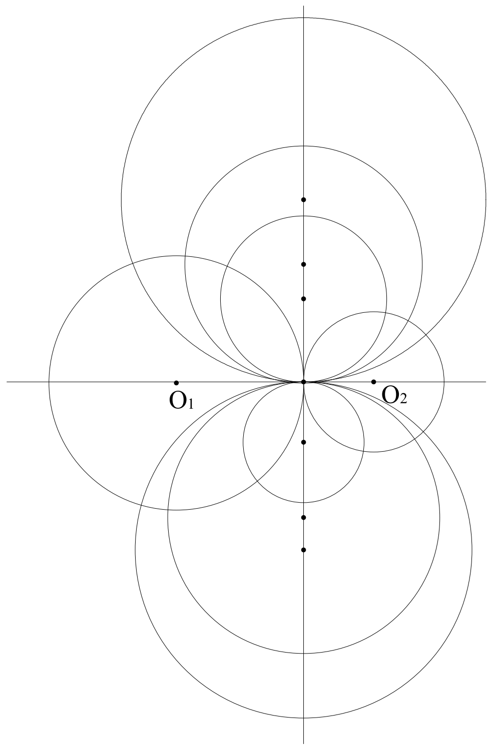


Figure C-5: A Tangent Coaxial Pencil Orthogonal to Two Given Tangent Circles

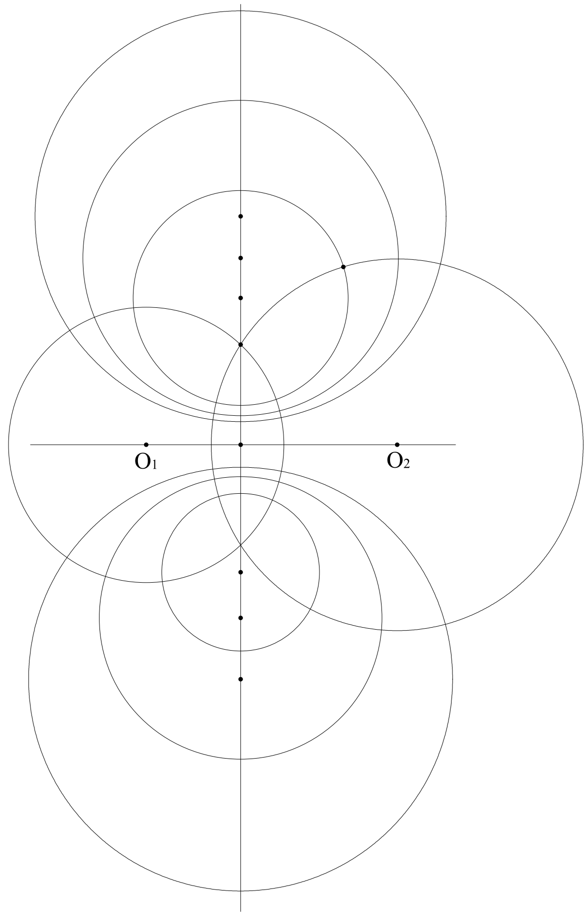


Figure C-6: An Intersecting Coaxial Pencil Orthogonal to Two Given Intersecting Circles

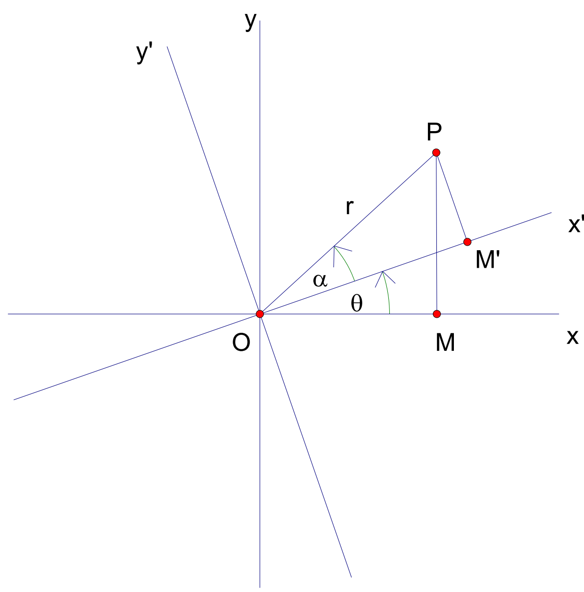
Appendix D

Rotation of Axes

The rotation of axes is a transformation of the plane that allows one to transform a quadratic form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{1}$$

to an equation without the xy-term. That is, a rotation can be found that forces the coefficient B to be 0.



In the figure above, suppose we rotate the axes through an angle θ to get a new set of axes, the $x'y'$-axes. A point P in the plane can then be described with coordinates from either set of axes. If P has coordinates (x, y) in the xy-plane, we see that

$$x = r\cos(\alpha + \theta) \tag{2}$$
$$y = r\sin(\alpha + \theta) \tag{3}$$

where θ is the angle of rotation and α is the angle OP makes with the x'-axis. From (2), we have that

$$\begin{aligned} x &= r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \\ &= x'\cos\theta - y'\sin\theta \end{aligned}$$

and from (3), we have

$$\begin{aligned} y &= r\sin\alpha\cos\theta + r\cos\alpha\sin\theta \\ &= y'\cos\theta + x'\sin\theta \\ &= x'\sin\theta + y'\cos\theta. \end{aligned}$$

If in equation (1) we replace x with $x'\cos\theta - y'\sin\theta$ and y with $x'\sin\theta + y'\cos\theta$, we can determine a value of θ which will make the coefficient B' of the $x'y'$-term equal to zero.

$$A(x'\cos\theta - y'\sin\theta)^2 + B(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + C(x'\sin\theta + \\ +y'\cos\theta)^2 + D(x'\cos\theta - y'\sin\theta) + E(x'\sin\theta + y'\cos\theta) + F = 0.$$

Only the first three terms

$$A(x'\cos\theta - y'\sin\theta)^2 + B(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + C(x'\sin\theta + y'\cos\theta)^2$$

contribute to an $x'y'$-term, so we will confine our attention to them. Multiplying out these terms gives

$$\begin{gathered} A(x'^2\cos^2\theta - 2x'y'\cos\theta\sin\theta + y'^2\sin^2\theta) + \\ +B(x'^2\cos\theta\sin\theta + x'y'\cos^2\theta - x'y'\sin^2\theta - y'^2\sin\theta\cos\theta) \\ +C(x'^2\sin^2\theta + 2x'y'\sin\theta\cos\theta + y'^2\cos^2\theta). \end{gathered}$$

We now determine the coefficient of the $x'y'$-term, which is

$$-2A\cos\theta\sin\theta + B(\cos^2\theta - \sin^2\theta) + 2C\sin\theta\cos\theta$$

which we could write as

$$-A\sin 2\theta + B\cos 2\theta + C\sin 2\theta.$$

If we set this equal to zero, we have

$$B\cos 2\theta = (A - C)\sin 2\theta.$$

Thus,

$$\cot 2\theta = \frac{A - C}{B}.$$

We choose the cotangent function rather than the tangent function since we know $B \neq 0$, for if it were, we would have no reason to rotate the axes. Therefore if we choose θ so that $\cot 2\theta = \frac{A-C}{B}$, then we will have an equation in our new coordinate system of the form

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0.$$

Example. Find the angle of rotation θ to remove the xy-term in the equation

$$11x^2 + 10\sqrt{3}xy + y^2 - 16 = 0.$$

Solution. For this equation $A = 11, B = 10\sqrt{3}$ and $C = 1$. Thus,

$$\cos 2\theta = \frac{A - C}{B} = \frac{11 - 1}{10\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Hence, $2\theta = \frac{\pi}{3}$ and $\theta = \frac{\pi}{6}$. Therefore, we let

$$\begin{aligned} x &= \frac{\sqrt{3}}{2}x' - \frac{1}{2}y' \\ y &= \frac{1}{2}x' + \frac{\sqrt{3}}{2}y' \end{aligned}$$

in our given equation.

$$11\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)^2 + 10\sqrt{3}\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right) + \left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right)^2 - 16 = 0.$$

Multiplying out and collecting terms we have

$$\left(\frac{33}{4} + \frac{30}{4} + \frac{1}{4}\right)x'^2 + \left(-\frac{11\sqrt{3}}{2} + \frac{10\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)x'y' + \left(\frac{11}{4} - \frac{30}{4} + \frac{3}{4}\right)y'^2 - 16 = 0.$$

This reduces to

$$16x'^2 - 4y'^2 - 16 = 0$$

which we recognize as the hyperbola

$$x'^2 - \frac{y'^2}{4} = 1.$$

Theorem. For the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

the following hold:

(i) If $B^2 - 4AC < 0$, then the equation represents an ellipse, a circle, a point, or else has no graph.

(ii) If $B^2 - 4AC > 0$, then the equation represents a hyperbola or a pair of intersecting lines.

(iii) If $B^2 - 4AC = 0$, then the equation represents a parabola, a line, a pair of parallel lines, or else has no graph.

Example. Use the above theorem to determine what the graph of the following equation might represent then find the rotation that will eliminate the xy-term in the following equation.

$$73x^2 + 72xy + 52y^2 + 30x - 40y - 75 = 0.$$

Solution Since $A = 73,\ B = 72,\ C = 52$, we have that $B^2 - 4AC < 0$ and we would expect to get an ellipse.

Now since $\frac{A-C}{B} = \frac{7}{24}$ we see that $\cot 2\theta = \frac{7}{24}$. For $0 < \theta < \frac{\pi}{2}$ and $\cot 2\theta = \frac{7}{24}$ we see that $\cos 2\theta = \frac{7}{25}$. Thus $\sin\theta = \sqrt{\dfrac{1-\frac{7}{25}}{2}} = \dfrac{3}{5}$ and $\cos\theta = \sqrt{\dfrac{1+\frac{7}{25}}{2}} = \dfrac{4}{5}$. The rotation equations are then

$$\begin{aligned} x &= \frac{4}{5}x' - \frac{3}{5}y' \\ y &= \frac{3}{5}x' + \frac{4}{5}y' \end{aligned}$$

Substituting into the given equation gives

$$\begin{aligned} 73(\frac{4}{5}x' &- \frac{3}{5}y')^2 + 72(\frac{4}{5}x' - \frac{3}{5}y')(\frac{3}{5}x' + \frac{4}{5}y') + 52(\frac{3}{5}x' + \frac{4}{5}y')^2 \\ &+ 30(\frac{4}{5}x' - \frac{3}{5}y') - 40(\frac{3}{5}x' + \frac{4}{5}y') - 75 = 0. \end{aligned}$$

This simplifies to

$$4x'^2 + y'^2 - 2y' = 3.$$

Completing the square gives

$$4x'^2 + (y'-1)^2 = 4 \quad \text{or } x'^2 + \frac{(y'-1)^2}{4} = 1$$

which we recognize as an ellipse whose center is $(0, 1)$ in the $x'y'-$coordinates.

Suppose a rotation of $\theta = \frac{\pi}{4}$ produces the following equation

$$y'^{\,2} + 4y' - 8x' + 12 = 0. \tag{4}$$

What was the original equation in the xy-plane?

This problem may be solved if the transformation given by

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta && (5) \\ y &= x'\sin\theta + y'\cos\theta && (6) \end{aligned}$$

has an inverse. In other words, can we solve for x' and y' in terms of x and y? To this end multiply equation (5) by $\cos\theta$ and equation (6) by $\sin\theta$ and add.

The result is $x\cos\theta + y\sin\theta = x'$. Now if we multiply equation (5) by $-\sin\theta$ and equation (6) by $\cos\theta$ and add, we get $-x\sin\theta + y\cos\theta = y'$. Hence the transformation

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}$$

is the inverse transformation and we should obtain the original equation if we replace x' and y' by

$$\begin{aligned} x' &= x\cos\frac{\pi}{4} + y\sin\frac{\pi}{4} \\ y' &= -x\sin\frac{\pi}{4} + y\cos\frac{\pi}{4} \end{aligned}$$

in equation (4). That is by,

$$\begin{aligned} x' &= \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y = \frac{\sqrt{2}}{2}(x+y) \\ y' &= -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y = \frac{\sqrt{2}}{2}(-x+y) \end{aligned}$$

This transformation should give the equation

$$x^2 - 2xy + y^2 - 12\sqrt{2}x - 4\sqrt{2}y + 24 = 0.$$

Notice that in this form the equation does not look like an equation for a parabola, which we see from equation (4).

Hints for Selected Exercises

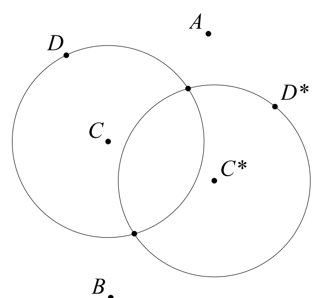
Hints for Selected Exercises

This appendix contains hints to some selected exercises. These hints should only be used after all other efforts have been exhausted. Since many problems in mathematics have more that one method of solution, these hints lead to one possible solution for the exercise noted. More hints will given be for early exercises than for the later exercises. It is assumed that, as the course progresses, one will become more adept at problem solving.

This appendix does not mean you cannot ask questions in class or my office about the exercises.

Chapter 1

Exercise 1.1.3 The points of intersection can be determined if a circle with center C^* and radius $C^*D^* = CD$ can be found on the opposite side of the line determined by AB.



Exercise 1.3.1 In Figure 1.3.9, one needs to show that $\angle CBD = \angle CAB$. Note that extra lines are in Figure 1.3.9 that were not in Figure 1.3.8. Drawing figures and including auxiliary lines are often very helpful in solving a problem in mathematics. Are there other angles in the Figure 1.3.9 equal to angle CAB?

Exercise 1.3.2 Products can come from ratios. Draw auxiliary lines to create similar triangles involving sides PA, PB, PC, and PD.

Exercise 1.3.5 By the Extended Law of Sines, $\dfrac{\sin A}{a} = \dfrac{1}{2R}$ and one of the area formulas is $|ABC| = \dfrac{1}{2}bc\sin A$.

Exercise 1.3.6 Find similar triangles and note that $|ABC| = \dfrac{1}{2}c \cdot CE$.

Exercise 1.3.11 From the previous problems, there are two ways this result can be obtained.

Exercise 1.3.12 Identify cyclic quadrilaterals which can be used to prove that $\angle PFE = \angle PFD$

Exercise 1.4.6 There are actually four such circles. See the figure on the next page. The centers of these circles are where angle bisectors meet. For experience, the interior circle and at least one of the exterior circles should be constructed.

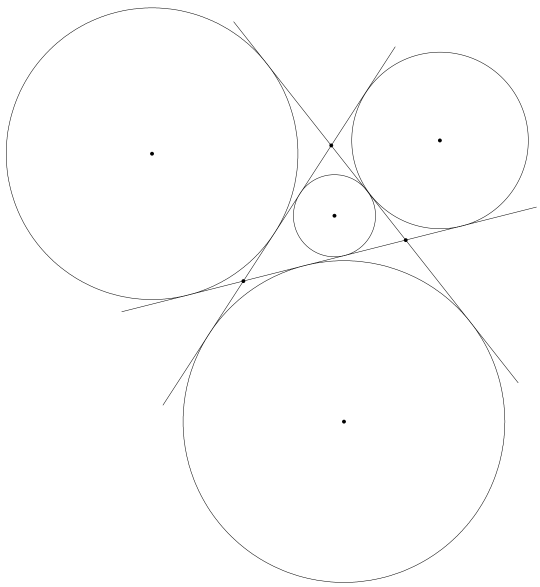


Figure for Exercise 1.4.6 Hint

Exercise 1.8.2 The properties of tangents to circles are crucial here. For example, note that $AD = AE$ and $CE = CF$.

Exercise 2.2.3 In the figure below BL is the bisector of the exterior angle at A. Note the relationship between the interior angle bisector and the exterior angle bisector at A. Be aware of the direction of angles.

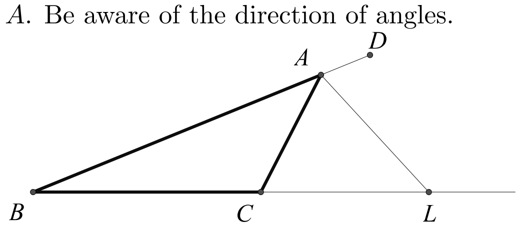


Figure for Exercise 2.2.3 Hint

Exercise 2.2.6 Use the results of Exercise 2.2.5.

Exercise 2.2.7 If D and E are the trisection points, then it must be shown that $CD^2 + CE^2 = \frac{5}{9}AB^2$. It appears that Stewart's Theorem might be useful.

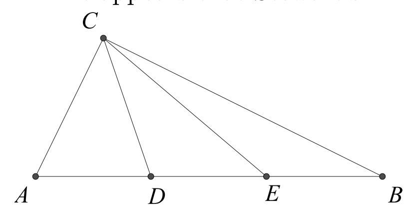


Figure for Exercise 2.2.7 Hint

Exercise 2.3.3 AD and PD are the common cevian lines for triangles ABC and PBC. See figure below. Drop perpendiculars from A and P to base BC.

Figure for Exercise 2.3.3 Hint

Exercise 2.3.5 There is not just one way to solve this problem. If all else fails, use Exercise 1.8.2.

Exercise 2.3.9 In the figure below, show that $BD = D'C$ and that this implies that $D'M_1 = M_1D$.

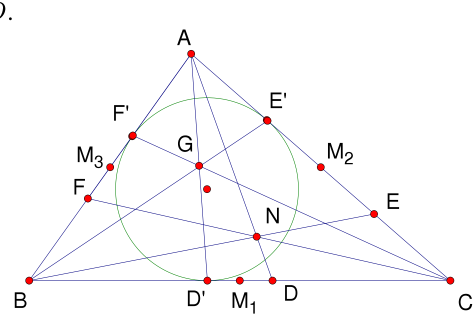


Figure for Exercise 2.3.9 Hint

Exercise 2.5.6 In the figure below, let A', B', C', D', P be the points of tangency for the lines a, b, c, d, p. Use Theorem 2.5.5 and the properties of angles and circles.

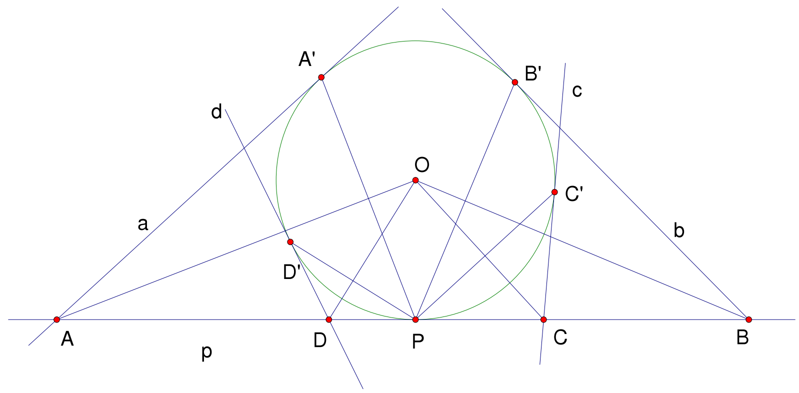


Figure for Exercise 2.5.5 Hint

Exercise 2.6.5 Use Menelaus's and Ceva's theorems.

Exercise 2.6.8 Use the figure below.

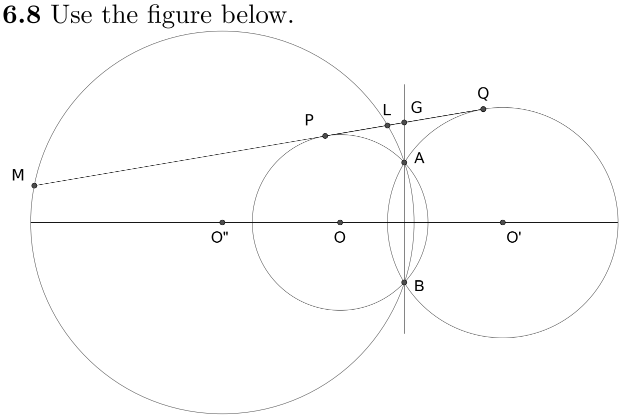


Figure for Exercise 2.6.8 Hint

Exercise 2.7.4 Let P be the center of the circle on AMO and let Q be the center of the circle on BMO. Show that $\angle POQ = 90°$; that is, the radius of the circle AMO is perpendicular to the radius of circle BMO at their point of intersection M.

Exercise 2.8.5 The radical axis must be a line perpendicular to the line of centers of the point and the circle. Find a point that has equal powers with respect to the point-circle and the circle.

Exercise 2.8.6 Find the radical center of the point and the two given circles.

Exercise 3.1.2 Read Appendix D.

Exercise 3.1.10 Draw a figure. Note similar triangles.

Exercise 3.2.2 Use Menelaus's theorem.

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