DE: Project 5: Solutions

1. Solve the IVP: $2y'' - 3y' + y = 0$

$y(0) = 2, \quad y'(0) = 1/2.$

Characteristic eq'n: $2r^2 - 3r + 1 = 0$

$$2r^2 - 2r - r + 1 = 0$$

$$2r(r-1) - 1(r-1) = 0$$

$$(2r-1)(r-1) = 0$$

$$r = 1/2, \ 1$$

general sol'n is $y = C_1 e^{1/2 t} + C_2 e^{t}$

$$y' = \tfrac{1}{2} C_1 e^{1/2 t} + C_2 e^{t}$$

Input initial data: $y(0) = C_1 + C_2 = 2$

$$y'(0) = \tfrac{1}{2} C_1 + C_2 = 1/2$$

$$1/2\, C_1 = 3/2 \Rightarrow C_1 = 3$$

$$\Rightarrow C_2 = -1$$

Thus, the particular sol'n is $\boxed{y = 3 e^{1/2 t} - e^{t}}$

2. The DE $ay'' + by' + cy = 0$ has characteristic eq'n

$$ar^2 + br + c = 0$$

which has root(s)

$$r = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Parts d) and e) are easiest:

$$r_1 = r_2 \text{ iff } b^2 - 4ac = 0$$

and r_1, r_2 are complex conjugates iff $b^2 - 4ac < 0$.

2. cont'd. For the rest, assume $a > 0$, and $b^2 - 4ac > 0$.
If $a < 0$, multiply the entire eq'n by -1 to change it.

For the roots to both be negative, we need $b > 0$
and
$$|b| > \sqrt{b^2 - 4ac}$$
$$\sqrt{b^2} > \sqrt{b^2 - 4ac}$$
since both radicands are positive, this implies:
$$b^2 > b^2 - 4ac$$
$$\Rightarrow \quad 0 > -4ac$$
$$\Rightarrow \quad ac > 0$$
Since a was assumed to be positive, c must also be positive.

a. { Thus, for both roots to be negative, we need
$$a, b, c > 0 \quad \text{and} \quad b = |b| > \sqrt{b^2 - 4ac}.$$
Alternatively, $a, b, c < 0$ and $|b| > \sqrt{b^2 - 4ac}$.

To have opposite signed roots, b can be of either sign, ~~and~~ but $\sqrt{b^2 - 4ac}$ must be larger than $|b|$.
If we take $b > 0$, then
$$\sqrt{b^2} < \sqrt{b^2 - 4ac}$$
$$b^2 < b^2 - 4ac$$
$$0 < -4ac$$
$$ac < 0$$
Again, since $a > 0$, this implies $c < 0$. Thus, the

b. { criteria is: $a, b > 0$, $c < 0$, $\sqrt{b^2 - 4ac} > |b| = b$
Alternatively, $a, b < 0$, $c > 0$, $\sqrt{b^2 - 4ac} > |b|$

c. { Finally, the roots are positive if $a > 0$, $c, b < 0$, $|b| > \sqrt{b^2 - 4ac}$
or $a < 0$, $c, b > 0$, $b = |b| > \sqrt{b^2 - 4ac}$ 2

3\. a.) $e^{2-\frac{\pi}{2}i} = e^2\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

$= e^2\left(0 + i(-1)\right) = \boxed{-ie^2}$

b.) $2^{1-i} = \left(e^{\ln 2}\right)^{1-i} = e^{\ln 2 - i\ln 2} = e^{\ln 2}e^{-i\ln 2}$

$= e^{\ln 2}\left(\cos(-\ln 2) + i\sin(-\ln 2)\right)$

$= \boxed{2\cos(\ln 2) - 2i\sin(\ln 2)}$

4\. $\begin{cases} 3u'' - u' + 2u = 0 \\ u(0) = 2, \ u'(0) = 0 \end{cases}$

Char. Eq'n: $3r^2 - r + 2 = 0 \qquad r = \frac{1}{6} \pm \frac{\sqrt{1-24}}{6} = \frac{1}{6} \pm i\frac{\sqrt{23}}{6}$

Sol'n is: $u = e^{t/6}\left(c_1\cos\left(\frac{\sqrt{23}}{6}t\right) + c_2\sin\left(\frac{\sqrt{23}}{6}t\right)\right)$

$u' = C_1\left(\frac{1}{6}e^{t/6}\cos\frac{\sqrt{23}}{6}t - \frac{\sqrt{23}}{6}e^{t/6}\sin\frac{\sqrt{23}}{6}t\right) + C_2\left(\frac{1}{6}e^{t/6}\sin\frac{\sqrt{23}}{6}t + \frac{\sqrt{23}}{6}e^{t/6}\cos\frac{\sqrt{23}}{6}t\right)$

$u(0) = C_1 = 2$

$u'(0) = \frac{1}{6}C_1 + \frac{\sqrt{23}}{6}C_2 = \frac{1}{3} + \frac{\sqrt{23}}{6}C_2 = 0$

$C_2 = \frac{-2}{\sqrt{23}}$

So the particular sol'n is:

$$\boxed{u(t) = e^{t/6}\left(2\cos\frac{\sqrt{23}}{6}t - \frac{2}{\sqrt{23}}\sin\frac{\sqrt{23}}{6}t\right)}$$

$|u(t)| = 10 \Leftrightarrow (u(t))^2 = 100$

$(u(t))^2 = e^{t/3}\left(2\cos\frac{\sqrt{23}}{6}t - \frac{2}{\sqrt{23}}\sin\frac{\sqrt{23}}{6}t\right)^2$

$= e^{t/3}\left[4\cos^2\frac{\sqrt{23}}{6}t + \frac{4}{23}\sin^2\frac{\sqrt{23}}{6}t - \frac{8}{\sqrt{23}}\cos\frac{\sqrt{23}}{6}t\sin\frac{\sqrt{23}}{6}t\right]$

This sucks...

By Wolfram Alpha $t \approx 6.409$