DE: Project 3: Solutions

1. a)

$$\frac{du}{dt} = -\alpha u^4$$

$$\int \frac{du}{u^4} = -\int \alpha \, dt$$

$$-\frac{1}{3u^3} = -\alpha t + C$$

$$3u^3 = \frac{1}{\alpha t + C}$$

$$u(t) = \sqrt[3]{\frac{1}{3\alpha t + C}}$$

$$u(0) = 2000 \Rightarrow 2000 = \sqrt[3]{\frac{1}{C}}$$

or $C = \frac{1}{2000^3} = \frac{1}{8{,}000{,}000{,}000}$

We can write $\frac{1}{3\alpha t + c} = \frac{(1/c)}{\frac{3\alpha}{c} t + 1}$

Using $\alpha = 2 \times 10^{-12}$ and $C = \frac{1}{8 \times 10^{-9}}$

$$\alpha / c = \frac{2 \times 10^{-12}}{8 \times 10^{-9}} = 0.25 \times 10^{-3} = 2.5 \times 10^{-4}$$

and $3 \frac{\alpha}{c} = 7.5 \times 10^{-4}$

The sol'n is then $u(t) = \frac{2000}{\sqrt[3]{1 + (7.5 \times 10^{-4}) t}}$

or

$$\boxed{u(t) = 2000 \left[1 + (7.5 \times 10^{-4}) t \right]^{-1/3}}$$

b) $u(\tau) = 600 = 2000 \left[1 + (7.5 \times 10^{-4}) \tau \right]^{-1/3}$

$$\frac{3}{10} = \left[1 + (7.5 \times 10^{-4}) \tau \right]^{-1/3}$$

$$\frac{10}{3} = \left[1 + (7.5 \times 10^{-4}) \tau \right]^{1/3}$$

$$\frac{1000}{27} = 1 + (7.5 \times 10^{-4}) \tau$$

$$\frac{973}{27} = (7.5 \times 10^{-4}) \tau$$

$$\tau = \frac{973}{27 (7.5 \times 10^{-4})}$$

$$\boxed{\tau \approx 48{,}049.38 \text{ s}}$$

2. $p' = rp$

$p(t) = p_0 e^{rt}$ $\quad p(t) = p_0 \left(e^{\ln 2}\right)^{\frac{t}{7}} = p_0\, 2^{t/7}$ w/out predators.
(not the question...)

$r = \frac{\ln 2}{7}$ where 7 is doubling-time in days

With predators: $\underbrace{p' = rp - 20{,}000}_{\text{solve this DE}} = \frac{\ln 2}{7} p - 20{,}000$

$$p' - rp = -20{,}000$$

$$\mu = e^{-rt} \qquad g = -20{,}000$$

General sol'n is: $p(t) = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)\, g(s)\, ds + \frac{C}{\mu(t)}$

$$= -e^{rt} \int_{t_0}^{t} e^{-rs}\, 20{,}000\, ds + Ce^{rt}$$

$$= -e^{+rt} \left(-\frac{1}{r}\, 20{,}000\, e^{-rs} \right)\Big|_{t_0}^{t} + Ce^{rt}$$

$$= \frac{20{,}000 \cdot 7}{\ln 2} + Ce^{\ln 2 \frac{t}{7}}$$

$$= \frac{140{,}000}{\ln 2} + C\, 2^{t/7}$$

$$p(0) = 200{,}000 = \frac{140{,}000}{\ln 2} + C$$

$$\text{So } C = 200{,}000 - \frac{140{,}000}{\ln 2}$$

and the sol'n is:

$$\boxed{p(t) = \left(200{,}000 - \frac{140{,}000}{\ln 2}\right) 2^{t/7} + \frac{140{,}000}{\ln 2}}$$

3. a) for $n=0$: $y' + p(t)y = q(t)$

We know the sol'n is

$$\boxed{\begin{array}{l} y = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)\, q(s)\, ds + \frac{C}{\mu(t)} \\ \text{where } \mu(t) = e^{\int p(t)dt} \end{array}}$$

b) for $n=1$: $y' + p(t)y = q(t)y \Rightarrow y' = y(q(t) - p(t))$

This is a separable DE! The sol'n is

$$\boxed{y = Ce^{\int q(t)-p(t)\, dt}}$$

c) To sub in $v = y^{1-n}$ we must find v'.

$$\frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = (1-n)\, y^{-n} \cdot y'$$

so $y' = \frac{v'}{(1-n)} y^n$

and $y = v y^n$

The eq'n becomes

$$\frac{v'}{(1-n)} \cancel{y^n} + p(t) v \cancel{y^n} = q(t) \cancel{y^n}$$

or

$$\boxed{v' + (1-n)p(t)\, v = (1-n)q(t)}$$ a linear DE in v.

d) when $n=2$: $v' - p(t) v = -q(t)$

$$\mu = e^{-\int p(t)dt}$$

and

$$v = \frac{-1}{\mu(t)} \int_{t_0}^{t} \mu(s) q(s)\, ds + \frac{C}{\mu(t)}$$

but $v = y^{1-n} = y^{-1}$, so the sol'n is

$$y^{-1} = -\frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s) g(s)\, ds + \frac{C}{\mu(t)}$$

or

$$\boxed{y = \left(-\frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)\, g(s)\, ds + \frac{C}{\mu(t)} \right)^{-1}}$$

4. $y' = y_1' + y_2'$ so

$$y' + p(t)\, y = y_1' + y_2' + p(t)\, y_1 + p(t)\, y_2$$

$$= \underbrace{(y_1' + p(t) y_1)}_{=0} + \underbrace{(y_2' + p(t) y_2)}_{=g(t)}$$

$$= g(t) \quad \blacksquare$$

5.

a) We know the general sol'n of (6) is:

$$y(t) = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s) g(s)\, ds + C \frac{1}{\mu(t)}$$

so $y_1(t) = 1/\mu(t)$ and $y_2(t) = 1/\mu(t) \int_{t_0}^{t} \mu(s)\, g(s)\, ds$

b) Recall $\mu(t) = e^{\int p(t) dt}$ so $\frac{1}{\mu(t)} = e^{-\int p(t) dt} = y_1(t)$

$$y_1'(t) = -p(t)\, e^{-\int p(t) dt} = -p(t)\, y_1(t) \quad \blacksquare$$

c) $y_2'(t) = \left[e^{-\int p(t)dt} \int_{t_0}^{t} \mu(s)\, g(s)\, ds \right]'$

$$= \underbrace{-p(t)\, e^{-\int p(t)dt} \int_{t_0}^{t} \mu(s)\, g(s)\, ds}_{-p(t)\, y_2(t)} + e^{-\int p(t)dt}\, \mu(t)\, g(t)$$

$$-p(t)\, y_2(t) \quad + \quad \frac{1}{\cancel{\mu(t)}} \cancel{\mu(t)}\, g(t)$$

$$= -p(t)\, y_2(t) + g(t) \quad \blacksquare$$

6. $\left.\begin{array}{l} \frac{dy}{dt} = -y^3 \\ y(0) = y_0 \end{array}\right\}$

$$y^{-3}\, dy = -dt$$
$$-\frac{1}{2} y^{-2} = -t + C$$
$$y^{-2} = 2t + C$$
$$y = \frac{1}{\sqrt{2t + C}}$$

$y(0) = \frac{1}{\sqrt{C}} = y_0$

so $C = \frac{1}{y_0^2}$

So

$$y = \frac{y_0}{\sqrt{2y_0^2 t + 1}}$$

domain of solution is $2y_0^2 t + 1 > 0$

$$2y_0^2 t > -1$$
$$\boxed{t > \frac{-1}{2y_0^2}}$$

7.

4.5

6. $\begin{cases} y' + y^3 = 0 \\ y(0) = y_0 \end{cases}$

$$\frac{dy}{y^3} = -1\, dt$$

$$-\frac{1}{2y^2} = -t + c$$

So $C = -\frac{1}{2y_0^2}$

and $\frac{1}{y^2} = 2t + \frac{1}{y_0^2}$

$$= \frac{2y_0^2 t + 1}{y_0^2}$$

$$y = \pm\sqrt{\frac{y_0^2}{2y_0^2 t + 1}}$$

Now, $2y_0^2 t + 1 > 0$

$$\boxed{t > -\frac{1}{2y_0^2}}$$

7. $(-e^{2x} - y + 1)\, dx + 1\, dy = 0$

$\partial_y M = -1$ $\quad p(x) = \frac{M_y - N_x}{N} = \frac{-1}{1} \Rightarrow \mu(x) = e^{\int p(x) dx} = e^{-x}$

$\partial_x N = 0$

New DE is:

$$(-e^x - ye^{-x} + e^{-x})\, dx + e^{-x}\, dy = 0$$

$$\psi(x,y) = \int -e^x - ye^{-x} + e^{-x}\, dx = -e^x + ye^{-x} - e^{-x} + h_1(y)$$

$$\psi(x,y) = \int e^{-x}\, dy = ye^{-x} + h_2(x)$$

$$\psi(x,y) = -e^x + ye^{-x} - e^{-x} = C$$

5

solving for y:

$$ye^{-x} = e^{x} + e^{-x} + C$$

$$\boxed{y = e^{2x} + 1 + Ce^{x}}$$

OMIT 8. OMIT (typo in assignment)