Differential Equations: Project 3

Due: Monday, 24 June 2013

Instructions: Complete all problems in a neat and organized fashion on your own paper. If you use Wolfram|Alpha, a calculator, or any other resources, please state what you used it for. You will not lose any points for doing so, as long as you're honest about how and why you used it.

1. Heat transfer from a body to its surroundings by radiation, based on the Stefan-Boltzmann law, is described by the differential equation

$$\frac{du}{dt} = -\alpha(u^4 - T^4), \tag{1}$$

where $u(t)$ is the absolute temperature of the body at time t, T is the absolute temperature of the surroundings, and α is a constant depending on the physical parameters of the body. However, if u is much larger than T, then solutions of equation (1) are well approximated by solutions of the simpler equation

$$\frac{du}{dt} = -\alpha u^4. \tag{2}$$

Suppose that a body with initial temperature $u_0 = 2000°K$ is surrounded by a medium with temperature $300°K$, and that $\alpha = 2.0 \times 10^{-12}\ °K^{-3}/s$.

a) Determine the approximate temperature of the body at any time by solving the DE given by equation (2). You may leave u_0 and α in your solution.

b) Find the time τ at which $u(\tau) = 600°K$; *i.e.,* twice the ambient temperature. Give your answer in scientific notation, correct to 3 significant figures. Up to this time, the (relative) error in using equation (2) to approximate the solutions of equation (1) is not more than 1%.

2. The population of mosquitoes in a certain area increases at a rate proportional to the current population, and in the absence of other factors, the population doubles every week. There are 200,000 mosquitoes in the area initially, and predators (birds, bats, *etc.*) eat 20,000 mosquitoes per day. Determine the population of mosquitoes in the area at any time (measured in days).

3. Bernoulli Equations. Sometimes it is possible to solve a nonlinear equation by making a change of the dependent variable that converts it into a linear equation. The most important such equations have the form

$$y' + p(t)y = q(t)y^n. \tag{3}$$

This is called a *Bernoulli equation*; named after Jakob Bernoulli.

a) Solve Bernoulli's equation (3) for $n = 0$.

b) Solve Bernoulli's equation (3) for $n = 1$.

c) Show that if $n \neq 0, 1$, then the substitution $v = y^{1-n}$ reduces Bernoulli's equation to a linear equation. This method was found by Leibniz in 1696.

The next two problems show that, given a solution y_2 of a non-homogeneous linear equation, a new solution can be constructed by adding a solution y_1 of the associated homogeneous equation. We will use this method to construct solutions to higher order linear DEs later in the semester.

4. Let $y = y_1(t)$ be a solution of the homogeneous equation

$$y' + p(t)y = 0, \tag{4}$$

and let $y = y_2(t)$ be a solution of the non-homogeneous equation

$$y' + p(t)y = g(t). \tag{5}$$

Show that $y = y_1(t) + y_2(t)$ is also a solution of equation (5).

5. *a*) Show that the solution of the general linear DE

$$y' + p(t)y = g(t) \tag{6}$$

can be written in the form

$$y = C\, y_1(t) + y_2(t), \tag{7}$$

where C is an arbitrary constant. Identify the functions y_1 and y_2.

b) Show that y_1 is a solution of the differential equation

$$y' + p(t)y = 0, \tag{8}$$

corresponding to $g(t) = 0$.

c) Show that y_2 is a solution of equation (6).

6. Solve the initial value problem and determine how the interval in which the solution exists depends on the initial data y_0.

$$\begin{cases} y' + y^3 = 0, \\ y(0) = y_0 \end{cases} \tag{9}$$

7. Find an integrating factor and solve the DE:

$$y' = e^{2x} + y - 1 \tag{10}$$

8. Solve the DE

$$(3x^2y + y^2) + (x^2 + xy)\, y' = 0 \tag{11}$$

using the integrating factor

$$\mu(x, y) = \frac{1}{xy(2x + y)}. \tag{12}$$

In class we solved this DE by using a different integrating factor. Verify that the solution you just obtained matches the one we got in class.