DE: Project 2 : Solutions

1. Volume is decreasing at a rate prop. to Area.

$V = \frac{4}{3}\pi r^3$, $A = 4\pi r^2$

Thus $\frac{dV}{dt} = -\gamma A$ where $\gamma > 0$ is the proportion constant.

but we want this in terms of V.

Notice that $V^{2/3} = \left(\frac{4}{3}\pi r^3\right)^{2/3} = \left(\frac{4}{3}\pi\right)^{2/3} r^2 = \frac{A}{k}$

for some constant $k > 0$.

Subbing in we get $\frac{dV}{dt} = -\gamma k V^{2/3}$

Combining constants, we can write $\boxed{\frac{dV}{dt} = -\alpha V^{2/3}}$

where $\alpha = \gamma k$.

2. a) $\frac{dy}{dt} = ay \Rightarrow \frac{dy}{y} = a\,dt \Rightarrow \ln y = at + C$

$\Rightarrow y = Ce^{at}$

So $\boxed{y_1(t) = Ce^{at}}$

b) Let $y = Ce^{at} + k$.

Then $dy/dt = Cae^{at}$ and ~~the~~ plugging into the DE we get

$\cancel{Cae^{at}} = \cancel{Cae^{at}} + ak - b$

$\Rightarrow k = b/a$

Thus our solution is $y = y_1(t) + b/a = \boxed{Ce^{at} + b/a}$

c) $\frac{dy}{dt} = ay - b = a\left(y - \frac{b}{a}\right)$

$\frac{dy}{y - b/a} = a\,dt$

$\ln\left(y - \frac{b}{a}\right) = at + C$

$y - b/a = Ce^{at}$

$\boxed{y = Ce^{at} + b/a}$ the same!

3. First solve $dy/dt = -ay \Rightarrow \frac{dy}{y} = -a\,dt$

$$\Rightarrow \ln y = -at + C$$
$$y = Ce^{-at}$$

Put $y_1(t) = Ce^{-at}$ and

$$y(t) = y_1(t) + k = Ce^{-at} + k$$
$$y' = -aCe^{-at}$$

and plugging into the DE we obtain

$$\cancel{-aCe^{-at}} = \cancel{-aCe^{-at}} - ak + b$$
$$\Rightarrow k = b/a$$

so the solution is $\boxed{y = Ce^{-at} + b/a}$

4. half-life $= T_2 = \frac{\ln 2}{r}$ so $r = \frac{\ln 2}{T_2} = \frac{\ln 2}{1620}$

To solve for quarter-life: $\frac{3}{4}\cancel{P_0} = \cancel{P_0}e^{-rt}$

$$\ln(3/4) = -rt$$
$$t = \frac{\ln(3/4)}{-r} = \frac{\ln(4/3)}{r}$$

but $r = \frac{\ln 2}{1620}$, so $\boxed{t = \frac{1620\ \ln(4/3)}{\ln 2} \approx 672.4}$

5. $du/dt = -0.15\,(u-10)$ $\quad u(0) = 70$

$$\frac{du}{u-10} = -0.15\,dt$$
$$\ln(u-10) = -0.15t + C$$
$$u - 10 = Ce^{-0.15t}$$
$$u = Ce^{-0.15t} + 10$$
$$u(0) = 70 \Rightarrow C = 60$$

so $\boxed{u(t) = 60e^{-0.15t} + 10}$

$$32 = 60e^{-0.15t} + 10$$
$$\frac{22}{60} = e^{-0.15t}$$
$$\ln\left(\frac{22}{60}\right) = -0.15t$$
$$t = \frac{\ln\left(\frac{22}{60}\right)}{-0.15}$$

$\boxed{t \approx 6.7}$ hours (?)

I guess...

6. $y_1(t) = e^{-3t}$ $\qquad$ $y_2(t) = e^{t}$

$y_1' = -3e^{-3t}$ $\qquad$ $y_2' = e^{t}$

$y_1'' = 9e^{-3t}$ $\qquad$ $y_2'' = e^{t}$

plug into DE:

$9e^{-3t} + 2(-3e^{-3t}) - 3e^{-3t}$ $\qquad$ $e^{t} + 2e^{t} - 3e^{t}$

$= (9-6-3)e^{-3t}$ $\qquad$ $= (1+2-3)e^{t}$

$= 0$ ✓ $\qquad$ $= 0$ ✓

7. a) $\mu(t) = e^{-2t}$ so the DE becomes

$$\frac{d}{dt}\left[e^{-2t}y\right] = t^2 \cancel{e^{2t}e^{-2t}}$$

or $d[e^{-2t}y] = t^2\,dt$

$$\Rightarrow e^{-2t}y = \frac{1}{3}t^3 + C$$

$$\Rightarrow \boxed{y = \frac{1}{3}t^3e^{2t} + Ce^{2t}}$$

b) $\mu = e^{\int \frac{2}{t}dt} = e^{\ln t^2} = t^2$

after rewriting: $ty' + 2y = \sin t$ as

$$y' + \frac{2}{t}y = \frac{\sin t}{t}$$

So the DE becomes:

$$d\left[t^2 y\right] = t\sin t\,dt$$

$$t^2 y = \int t\sin t\,dt$$

Integrate by Parts! Get

$$t^2 y = -t\cos t + \sin t + C$$

$$\boxed{y = -\frac{1}{t}\cos t + \frac{1}{t^2}\sin t + \frac{1}{t^2}C}$$

c) $\mu = e^{\frac{1}{2}t}$ $\Rightarrow$ DE becomes:

$$\frac{d}{dt}\left[e^{1/2t}y\right] = 3t^2e^{1/2t}$$

$$d\left[e^{1/2t}y\right] = 3t^2e^{1/2t}\,dt$$

$$e^{1/2t}y = \int 3t^2e^{1/2t}\,dt$$ IbP again! Woohoo!

	u	dv
$+$	t^2	$e^{1/2t}$
$-$	$2t$	$2e^{1/2t}$
$+$	2	$4e^{1/2t}$
	0	$8e^{1/2t}$

$$\Rightarrow 3\left(2t^2e^{1/2t} - 8te^{1/2t} + 16e^{1/2t}\right) + C$$

so we get:

$$e^{1/2t}y = 6t^2e^{1/2t} - 24te^{1/2t} + 48e^{1/2t} + C$$

and

$$\boxed{y = 6t^2 - 24t + 48 + Ce^{-1/2t}}$$