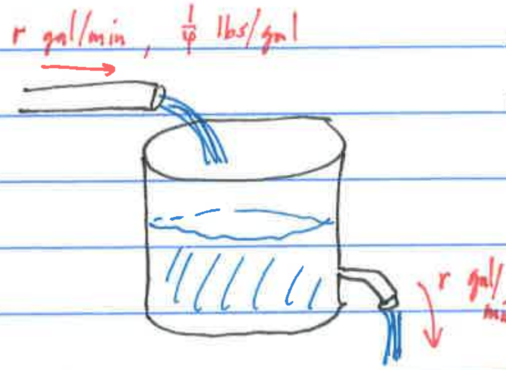
2.3: Modeling w/ First order ODE

In this section we'll use DE to write mathematical eq'ns that model physical phenomena.

Ex. Mixing Problem



A tank holds 100 gal of H_2O w/ Q_0 lbs of salt dissolved in it.
We pump in a mixture of salt water containing 1/4 lb/gal at a rate of r gal/min.

Simultaneously, water is drained from the bottom of the tank at a rate of r gal/min. Assume the water in the tank is mixed immediately.

We want to set up a DE, find the limiting amount of salt, find the time T after which Q(T) is within 2% of the limiting amount given certain initial conditions.

1. Writing the DE.

The change in salt amount is (concentration in)(rate in) − (concentration out)(rate out)

$$\frac{dQ}{dt} = \frac{1}{4} r - ?\, r$$

We need to determine the "concentration out" of the tank. There are $Q(t)$ lbs of salt in 100 gal of H_2O, so conc. out is $Q/100$ and the DE is

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100}$$

2. <u>Solving the DE.</u>

Write $\frac{dQ}{dt} + \frac{r}{100} Q = \frac{r}{4}$

This has integrating factor $\mu(t) = e^{rt/100}$, and general solution:

$$Q(t) = e^{-rt/100} \int_{t_0}^{t} \frac{r}{4} e^{rs/100}\, ds + C e^{-rt/100}$$

$$= e^{-rt} \left(\frac{100}{r} \cdot \frac{r}{4} e^{rs/100} \right) \Bigg|_{t_0}^{t} + C e^{-rt/100}$$

$$Q(t) = 25 + C e^{-rt/100}$$

We have the initial condition $Q(0) = Q_0$, so that

$$Q_0 = 25 + C \quad \text{or} \quad C = Q_0 - 25$$

The particular sol'n is then

$$Q(t) = (Q_0 - 25) e^{-rt/100} + 25$$

or $$\boxed{Q(t) = 25\left(1 - e^{-rt/100}\right) + Q_0 e^{-rt/100}}$$

3. <u>Determining the limiting amount.</u>

Now consider $t \to \infty$, we have

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty}\left[25\left(1-e^{-rt/100}\right) + Q_0 e^{-rt/100}\right]$$

$$= 25(1-0) + Q_0(0)$$

$$= 25$$

Thus $\boxed{Q_L = 25 \text{ lbs salt.}}$

4. <u>Solving an IVP.</u>

Suppose $r = 3$ and $Q_0 = 2Q_L = 50$.
Find T such that $Q(T) = \cancel{98}\ 102\%$ of Q_L. (within 2% of Q_L)

The sol'n $Q(t)$ is now given by $Q(t) = 25\left(1-e^{-3t/100}\right) + 50e^{-3t/100}$

We want $Q(T) = \frac{51}{50}(Q_L) = \frac{51}{2}$ so that

$$\frac{51}{2} = 25\left(1-e^{-3T/100}\right) + 50e^{-3T/100}$$

$$51/2 = 25 + 25e^{-3T/100}$$

$$+\frac{1}{2} = 25e^{-3T/100}$$

$$+\frac{1}{50} = e^{-3T/100}$$

$$-\ln(50) = \frac{-3T}{100}$$

so $\boxed{T = \frac{100}{3}\ln(50)} \approx 130.4$ min

5. Another kind of problem.

Suppose $Q_0 = 2Q_L$ again. Find r s.t. $T = 45$ min.

so $Q(T) = \frac{51}{2}$ again and we have

$$\frac{51}{2} = 25 + 25e^{-r\,45/100}$$

$$\left.\begin{aligned} \frac{1}{2} &= 25e^{-r\,45/100} \\ \frac{1}{50} &= e^{-r\,45/100} \\ -\ln 50 &= -r\,\frac{45}{100} \end{aligned}\right\} \Rightarrow r = \boxed{\frac{100}{45}\ln(50)}$$

$$\approx 8.69 \text{ gal/min}$$

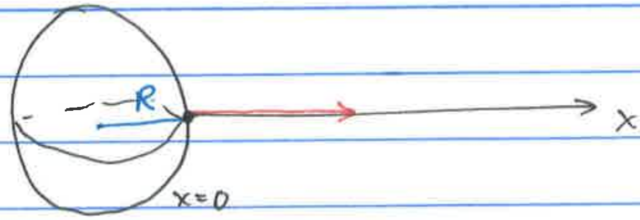
Ex. Escape velocity.

An object w/ mass m is projected away from the Earth, perpendicular to the surface, with initial velocity v_0.

Disregard air resistance, but consider gravity.

Find an expression for the velocity at any time t during this motion.

Also find the initial velocity v_0 required to lift the body to an altitude ξ above the earth's surface, and find the initial velocity at which the object will not return to the earth.



Newton's theory of gravity says that gravity will act on the object with a force inversely proportional to the objects' distance from the center of the earth.

$$w(x) = \frac{-k}{(x+R)^2}$$

On the surface of the earth $w(0) = -mg$, therefore $k = mgR^2$, and

$$w(x) = \frac{-mgR^2}{(x+R)^2}$$

The DE is thus $\quad a = \cancel{m}\frac{dv}{dt} = \frac{-\cancel{m}gR^2}{(x+R)^2}$

w/ initial condition: $\quad v(0) = v_0$.

Now this eqn has too many variables: v, x, t. We can eliminate t by using the chain rule (since this is 1-D motion).

$$\frac{dv}{dt} = \frac{dv}{dx}\underbrace{\frac{dx}{dt}}_{v} = v\frac{dv}{dx}$$

The DE becomes: $\boxed{v\frac{dv}{dx} = \frac{-gR^2}{(x+R)^2}}$ $\qquad (*)$

This is separable: $v\,dv = \dfrac{-gR^2}{(x+R)^2}\,dx$

The particular sol'n is given by:

$$\int_{v_0}^{v} t\,dt = \int_0^x \frac{-gR^2}{(t+R)^2}\,dt$$

$$\frac{1}{2}v^2 - \frac{1}{2}v_0^2 = \frac{gR^2}{x+R} - gR$$

solving for v:

$$v = \pm\sqrt{\frac{2gR^2}{x+R} + v_0^2 - 2gR}$$

For physical reasons, we must take the + sign when the object is moving away from Earth, and − when it's moving back toward Earth.

Set $v=0$ and $x=\xi$ to determine the maximum altitude the body reaches w/ a given initial velocity.

We get $0 = \sqrt{\dfrac{2gR^2}{\xi+R} + v_0^2 - 2gR}$

$$\frac{2gR^2}{\xi+R} = 2gR - v_0^2$$

$$\xi + R = \frac{2gR^2}{2gR - v_0^2}$$

$$\xi = \frac{2gR^2 - 2gR^2 + v_0^2 R}{2gR - v_0^2}$$

$$\boxed{\xi = \frac{v_0^2 R}{2gR - v_0^2}}$$

Solving (**) for v_0 we obtain

$$v_0^2 = \frac{2gR^2 - 2gR^2 + 2g\xi R}{\xi + R} = \frac{2gR\xi}{\xi + R}$$

$$\text{or} \qquad v_0 = \sqrt{\frac{2gR\xi}{\xi + R}}$$

We can find the escape velocity by taking the limit as $\xi \to \infty$. We obtain

$$\lim_{\xi \to \infty} v_0 = \lim_{\xi \to \infty} \sqrt{\frac{2gR\xi}{\xi + R}} = \boxed{\sqrt{2gR} = v_e}$$

g and R are known constants, so we can compute this for any object. We get

$$\boxed{v_e \approx 6.9 \ mi/s \ \text{ or } \ 11.1 \ km/s}$$

This neglects air resistance and other factors, so v_e is actually slightly higher than this.