Math 555: Differential Equations
Exam 3: Chapter 5

Friday, 26 July 2013

Name: KEY

Instructions: Complete all problems, showing all work. Problems are graded based not only on whether the answer is correct, but if the work leading up to the answer is correct. Simplify as necessary. Leave any answers involving π or irreducible square roots or logs in terms of such.

1. [20 points] Use the method of variation of parameters to find the general solution of the DE.

$$y'' + 2y' + y = 3e^{-t}$$

First solve the homog. DE: $y'' + 2y' + y = 0$

$$\left.\begin{array}{l} r^2 + r + 1 = 0 \\ (r+1)^2 = 0 \\ r = -1 \end{array}\right\} \quad \left.\begin{array}{l} y_1 = e^{-t} \\ y_2 = te^{-t} \end{array}\right\} \quad W(y_1, y_2) = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-2t} - te^{-2t} + te^{-2t} = e^{-2t}$$

From the original DE $g(t) = 3e^{-t}$

Now plug into the formula:

$$y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)\, g(s)}{W(s)}\, ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)\, g(s)}{W(s)}\, ds$$

$$= -e^{-t}\int_{t_0}^{t} \frac{(s\cancel{e^{-s}})(3\cancel{e^{-s}})}{\cancel{e^{-2s}}}\, ds + te^{-t}\int_{t_0}^{t} \frac{\cancel{e^{-s}}\,(3\cancel{e^{-s}})}{\cancel{e^{-2s}}}\, ds$$

$$= -e^{-t}\left(\frac{3}{2}s^2 \Big|_{t_0}^{t}\right) + te^{-t}\left(3s \Big|_{t_0}^{t}\right)$$

$$= -\frac{3}{2}t^2e^{-t} + C_1e^{-t} + 3t^2e^{-t} + C_2te^{-t}$$

$\Rightarrow$

$$\boxed{y(t) = C_1e^{-t} + C_2te^{-t} + \frac{3}{2}t^2e^{-t}}$$

2. [15 points] Find the radius of convergence and interval of convergence of the series.

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n\,2^n}$$

Ratio Test: $\left| \frac{(x+1)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n\,2^n}{(x+1)^n} \right| = |x+1| \left| \frac{n\,\cancel{2^n}}{(n+1)\cancel{2^n}\,2} \right| = \frac{|x+1|}{2} \cdot \frac{n}{n+1}$

$\lim_{n\to\infty} \frac{|x+1|}{2} \frac{n}{n+1} = \frac{|x+1|}{2} < 1 \Rightarrow |x+1| < 2 \Rightarrow \boxed{R=2}$

When $x=-3$: $\sum_{n=1}^{\infty} \frac{(-2)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges

when $x=1$ $\sum_{n=1}^{\infty} \frac{2^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Thus IC: $\boxed{-3 \le x < 1}$ or $[-3,1)$

3. [15 points] Find a power series representation for $f(x) = \ln(1-x)$ centered at $x_0 = 0$. What is the radius of convergence?

$(\ln(1-x))' = \frac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n$ by geometric series

Thus $\ln(1-x) = -\sum_{n=0}^{\infty} \int x^n\,dx = \boxed{-\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \ln(1-x)}$

and $\boxed{R=1}$ since it is geometric.

4. [20 points] Find power series solutions y_1 and y_2 of the DE.

$$y'' + xy' + 2y = 0, \quad x_0 = 0$$

Clearly label (put a box around) the recurrence relation. What is the value of the Wronksian $W(y_1, y_2)(x_0)$?

Assume: $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2}$

Plug in:

$$\sum_{n=2}^{\infty} (n-1) n a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\Rightarrow 2 \cdot 1\, a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + 2a_0 + \sum_{n=1}^{\infty} 2a_n x^n = 0$$

$$\Rightarrow (2 \cdot 1\, a_2 + 2a_0) + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n+2) a_n \right] x^n = 0$$

Thus $a_2 = \frac{-2}{2 \cdot 1} a_0 = -a_0$ and $\boxed{a_{n+2} = \frac{-1}{n+1} a_n, \ n \geq 1}$ Recurrence Relation

evens:

$n=2$: $a_4 = \frac{-1}{3} a_2 = \frac{1}{3 \cdot 1} a_0$

$n=4$: $a_6 = \frac{-1}{5} a_4 = \frac{-1}{5 \cdot 3 \cdot 1} a_0$

$n=6$: $a_8 = \frac{-1}{7} a_6 = \frac{1}{7 \cdot 5 \cdot 3 \cdot 1} a_0$

$n=2k-2$: $a_{2k} = \frac{(-1)^k\, 2 \cdot (k!)}{(2k+1)!} a_0$

$$\Rightarrow \boxed{y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2\,(n!)}{(2n+1)!} x^{2n}}$$

odds:

$n=1$: $a_3 = \frac{-1}{2} a_1$

$n=3$: $a_5 = \frac{-1}{4} a_3 = \frac{1}{4 \cdot 2} a_1$

$n=5$: $a_7 = \frac{-1}{6} a_5 = \frac{-1}{6 \cdot 4 \cdot 2} a_1$

$n=2k-1$: $a_{2k+1} = \frac{(-1)^{k+1}}{2\,(k!)} a_1$

$$\boxed{y_2 = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{2\,(n!)} x^{2n+1}}$$

$\boxed{W(y_1, y_2)(0) = 1 \text{ always for series sol'ns.}}$

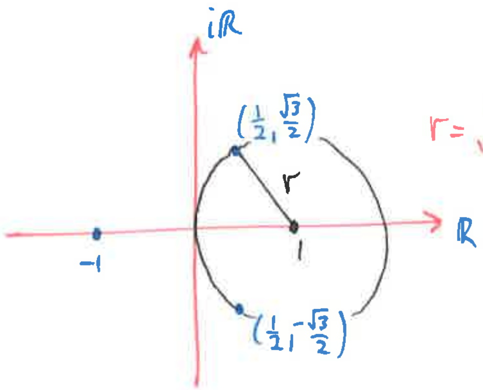
$$\Rightarrow \boxed{y = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n\, 2\,(n!)}{(2n+1)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2\,(n!)} x^{2n+1}}$$

5. [15 points] Determine a lower bound for the radius of convergence of series solutions to the DE, centered at $x_0 = 1$, *without solving the DE.*

$$(1 + x^3)y'' + 4xy' + y = 0$$

$1+x^3 = (1+x)(1 - x + x^2)$

$x=-1, \quad x = \dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2} i$



$r = \sqrt{(1-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$

Thus, the minimum radius of convergence of the sol'n is $\boxed{R=1}$

6. [15 points] Determine the general solution of the DE for $x > 0$.

$$x^2y'' + 3xy + 5y = 0$$

$r^2 + 2r + 5 = 0$

$r = \dfrac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$

$\Rightarrow \boxed{y = \frac{1}{x}\left(c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))\right)}$