Math 555: Differential Equations
Exam 2: Chapter 3

Friday, 12 July 2013

Name: KEY

Instructions: Complete all problems, showing all work. Problems are graded based not only on whether the answer is correct, but if the work leading up to the answer is correct. Simplify as necessary. Leave any answers involving π or irreducible square roots or logs in terms of such.

1. Solve the differential equations. Write the general solution in the form $y = c_1 y_1 + c_2 y_2$.

a.) $y'' + 2y' + 2y = 0$

$r^2 + 2r + 2 = 0$

$r = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$

$\lambda = -1, \ \mu = 1$

$\Rightarrow \boxed{y = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t}$

b.) $y'' + 3y' + 2y = 0$

$r^2 + 3r + 2 = 0$

$(r+1)(r+2) = 0$

$r = -1, -2$

$\Rightarrow \boxed{y = c_1 e^{-t} + c_2 e^{-2t}}$

c.) $9y'' - 12y' + 4y = 0$

$9r^2 - 12r + 4 = 0$

$(3r-2)^2 = 0$

$r = \frac{2}{3}$

$\Rightarrow \boxed{y = c_1 e^{2/3 t} + c_2 t e^{2/3 t}}$

2. Find the Wronskian of the two functions $y_1 = \cos t$ and $y_2 = \sin t$.

$$W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = \boxed{1}$$

~~**3.**~~ Find the Wronskian of the SODE *without* solving the equation.

$$(\cos t)y'' + (\sin t)y' - ty = 0$$

$$y'' + \tan t \, y' - \frac{t}{\cos t} y = 0$$

$\cos t$

$\cancel{\sin t} = u$

$-\sin t \, \cancel{\cos t} \, dt = du$

$$W = e^{-\int p(t)\,dt} = e^{-\int \tan t \, dt} = e^{-\int \frac{\sin t}{\cos t} dt} = e^{\int \frac{1}{u} du}$$

$$= e^{\ln u + C}$$

$$= Cu$$

$$= \boxed{C \cdot \cos t}$$

4. If $W(f, g) = t^2 e^t$ and $f(t) = t$, find $g(t)$.

$$W(f, g) = fg' - f'g$$

$$= tg' - g = t^2 e^t$$

$$g' - \frac{1}{t} g = te^t$$

$$\mu(t) = e^{-\int \frac{1}{t} dt} = e^{-\ln t} = \frac{1}{t}$$

$$g = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)\, g(s)\, ds + \frac{C}{\mu(t)}$$

$$g = t \int_{t_0}^{t} e^s \, ds + Ct = te^t + Ct$$

Put $C = 0$ to get

$$\boxed{g(t) = te^t}$$

~~5.~~ Solve the IVP: $y'' + 4y' + 4y = 0$, $y(-1) = 2$, $y'(-1) = 1$.

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$

$r = -2$

$y = C_1 e^{-2t} + C_2 t e^{-2t}$

$y' = -2C_1 e^{-2t} + C_2 e^{-2t} - 2C_2 t e^{-2t}$

$y(-1) = C_1 e^2 + C_2 e^2 = 2 \Rightarrow 2C_1 e^2 + 2C_2 e^2 = 4$

$y'(-1) = -2C_1 e^2 + C_2 e^2 + 2C_2 e^2 = 1 \quad -2C_1 e^2 + 3C_2 e^2 = 1$

$-2C_1 e^2 + 3C_2 e^2 = 1 \qquad 5C_2 e^2 = 5$

$\boxed{C_2 = e^{-2}}$

$\Rightarrow C_1 e^2 + 1 = 2$

$C_1 e^2 = 1$

$\boxed{C_1 = e^{-2}}$

Thus, the sol'n is:

$\boxed{y = e^{-2t-2} + t e^{-2t-2}}$

6. Determine the form of $Y(t)$ if the method of undetermined coefficients is to be used. *Do <u>not</u> solve for the coefficients.*

$$y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t \sin 2t$$

$(r-2)^2 = 0$

$\Rightarrow r = 2$

$y = C_1 e^{2t} + C_2 t e^{2t}$ is homog. sol'n.

Thus, our guess for Y must be:

$\boxed{Y(t) = At^2 + Bt^2 e^{2t} + Ct \sin 2t + Dt \cos 2t}$

7. Given that $y_1 = t^2$ is a solution of the SODE

$$t^2 y'' - 4ty' + 6y = 0, \quad t > 0$$

use the method of reduction of order to find another solution $y_2(t)$. Show that $\{y_1, y_2\}$ form a fundamental solution set, and write the general solution in the form $y = c_1 y_1 + c_2 y_2$.

Put $y = v t^2 = t^2 v$

$y' = v' t^2 + 2tv = t^2 v' + 2tv$

$y'' = v'' t^2 + 2tv' + 2tv' + 2v$

$= t^2 v'' + 4tv' + 2v$

Plug back in for y, y', y'' in SODE:

$$t^4 v'' + 4t^3 v' + 2t^2 v - 4t^3 v' - 8t^2 v + 6t^2 v = 0$$

$\Rightarrow t^4 v'' = 0$

but $t^4 \neq 0$ since $t \neq 0$, so $v'' = 0$

$\Rightarrow v' = C_2$

and $v = C_2 t + C_1$

thus $y = v t^2 = \underbrace{C_2 t^3}_{y_2?} + \underbrace{C_1 t^2}_{y_1}$ $\boxed{y_2 = t^3}$

$$W(t^2, t^3) = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = 3t^4 - 2t^4 = t^4 \neq 0 \quad \text{since } t \neq 0.$$

Thus $\{t^2, t^3\}$ form fund. sol'n set and $\boxed{y = C_1 t^2 + C_2 t^3}$ is the general solution of the SODE.

8. Use the method of undetermined coefficients to find the general solution of the SODE

$$y'' + 2y' + y = 2e^{-t} + \cos t$$

Write the answer in the form $y = Y + c_1y_1 + c_2y_2$.

homog. eq'n: $y'' + 2y' + y = 0$

$$(r+1)^2 = 0$$

$$r = -1$$

$$y = c_1 e^{-t} + c_2 t e^{-t}$$

Guess: $Y(t) = At^2e^{-t} + B\cos t + C\sin t$

$$Y'(t) = 2Ate^{-t} - At^2e^{-t} - B\sin t + C\cos t$$

$$Y''(t) = 2Ae^{-t} - 2Ate^{-t} - 2Ate^{-t} + At^2e^{-t} - B\cos t - C\sin t$$

$$= 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t} - B\cos t - C\sin t$$

Plugging into the DE:

$$2Ae^{-t} - \cancel{4Ate^{-t}} + \cancel{At^2e^{-t}} - \cancel{B\cos t} - \cancel{C\sin t} + \cancel{4Ate^{-t}} - \cancel{2At^2e^{-t}} - 2B\sin t + 2C\cos t +$$

$$\cancel{At^2e^{-t}} + \cancel{B\cos t} + \cancel{C\sin t} = 2e^{-t} + \cos t$$

Get some eq'ns:

$$2Ae^{-t} = 2e^{-t} \Rightarrow A = 1$$

$$-2B\sin t = 0 \Rightarrow B = 0$$

$$2C\cos t = \cos t \Rightarrow C = \frac{1}{2}$$

Thus $Y(t) = t^2e^{-t} + \frac{1}{2}\sin t$

and

$$\boxed{y = t^2e^{-t} + \frac{1}{2}\sin t + c_1e^{-t} + c_2te^{-t}}$$