Math 555: Differential Equations
Exam 1: Chapters 1 and 2

Friday, 28 June 2013

Name: KEY

Instructions: Complete all problems, showing all work. Problems are graded based not only on whether the answer is correct, but if the work leading up to the answer is correct. Simplify as necessary. Leave any answers involving π or irreducible square roots or logs in terms of such.

1. Verify that the given functions are solutions of the differential equation.

$$y'' - y' - 6y = 0\,;\quad y_1(t) = \bar{e}^{2t},\quad y_2(t) = e^{+3t}$$

y_1: $y' = -2e^{-2t}$

$y'' = 4e^{-2t}$

$y'' - y' - 6y = 4e^{-2t} + 2e^{-2t} - 6e^{-2t} = 0$ ✓

y_2: $y' = 3e^{3t}$

$y'' = 9e^{3t}$

$y'' - y' - 6y = 9e^{3t} - 3e^{3t} - 6e^{3t} = 0$ ✓

2. Determine the values of r for which the given differential equation has solutions of the form $y = e^{rt}$.

$$y'' + 2y' - 3y = 0$$

$y' = re^{rt}$

$y'' = r^2e^{rt}$

$y'' + 2y' - 3y = 0$ becomes

$r^2e^{rt} + 2re^{rt} - 3e^{rt} = 0$

$e^{rt}(r^2 + 2r - 3) = 0$

$r^2 + 2r - 3 = 0$

$(r+3)(r-1) = 0$

$\boxed{r = -3, 1}$

3. – 5. Find general solutions of the differential equations using any method.

3. $2y' + y = 3t$

$y' + \frac{1}{2}y = \frac{3}{2}t$

$y = e^{-\frac{1}{2}t}\int_{t_0}^{t} e^{\frac{1}{2}s}\cdot\frac{3}{2}s\, ds + Ce^{-1/2t}$

$y = e^{-1/2t}\cdot\frac{3}{2}\left(2se^{1/2s} - 4e^{1/2s}\right)\Big|_{t_0}^{t} + Ce^{-1/2t}$

$$\boxed{y = 3t - 6 + Ce^{-1/2t}}$$

MORE ROOM FOR WORK

4. $(2x + 3) + (2y - 2)y' = 0$

$\int(2x+3)\,dx + \int(2y-2)\,dy = 0$

$x^2 + 3x + y^2 - 2y = C$

$y^2 - 2y + \underbrace{x^2 + 3x + C}_{A} = 0$

$y = \frac{2 \pm \sqrt{4-4A}}{2} = \frac{2 \pm 2\sqrt{1-A}}{2} = 1 \pm \sqrt{1-A}$

so,

$y = 1 \pm \sqrt{1 - x^2 - 3x - C}$ or

$$\boxed{y = 1 \pm \sqrt{C - x^2 - 3x}}$$

5. $y' = x^2(1 + y^2)$

$\frac{dy}{1+y^2} = x^2\,dx$

$\arctan y = \frac{1}{3}x^3 + C$

$$\boxed{y = \tan\left(\frac{1}{3}x^3 + C\right)}$$

6. – 8. Solve the initial value problems using any method.

6. $y' = 2y^2 + xy^2, \quad y(0) = 1$

$$\int_1 \frac{dy}{y^2} = \int (2+x)\,dx$$

$$\int_1^y \frac{dt}{t^2} = \int_0^x (2+t)\,dt = -\frac{1}{t}\Big|_1^y = 2t + \frac{1}{2}t^2\Big|_0^x$$

$$-\frac{1}{y} + 1 = 2x + \frac{1}{2}x^2$$

$$-\frac{1}{y} = x + \frac{1}{2}x^2 - 1$$

$$\boxed{y = \frac{1}{1 - \frac{1}{2}x^2 - 2x}}$$

7. $t\,y' + 2\,y = \dfrac{\cos t}{t}, \quad y(\pi) = 0, \quad t > 0$

$$y' + \frac{2}{t}y = \frac{\cos t}{t^2}$$

$$\mu = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$$

soln is:

$$y = \frac{1}{t^2}\int_0^t s^2 \frac{\cos s}{s^2}\,ds + \frac{C}{t^2} = \frac{1}{t^2}\sin t + \frac{C}{t^2}$$

$$y(\pi) = 0 = \frac{1}{\pi^2}\sin\pi + \frac{C}{\pi^2} \quad (\sin\pi = 0)$$

$$\Rightarrow C = 0$$

thus, the soln is:

$$\boxed{y = \frac{1}{t^2}\sin t}$$

MORE ROOM FOR WORK

8. $(9x^2 + y - 1)\,dx + (4y + x)\,dy = 0, \quad y(1) = 0$

$M_y = 1$, $N_x = 1$ } exact!

$$\int M\,dx = \int 9x^2 + y - 1\,dx = 3x^3 + xy - x + h_1(y)$$

$$\int N\,dy = \int -4y + x\,dy = -2y^2 + xy + h_2(x)$$

Thus $\psi(x,y) = 3x^3 + xy - x - 2y^2$

Soln is:

$$3x^3 + xy - x - 2y^2 = C$$

Plugging in $(1,0)$:

$$3 + 0 - 1 - 0 = C$$

$$\Rightarrow C = 2$$

So,

$$\boxed{3x^3 + xy - x - 2y^2 = 2}$$

$$2y^2 - xy + (x - 3x^3 - 2) = 0$$

$$y = \frac{x \pm \sqrt{x^2 - 8(x - 3x^3 - 2)}}{4}$$

from I.C.:

$$\boxed{y = \frac{x - \sqrt{16 - 8x + x^2 + 24x^3}}{4}}$$

9. Determine the interval on which the solution of the initial value problem is guaranteed to exist *without* solving the DE.

$$(\ln t)y' + y = \cot t, \quad y(2) = 3$$

$$y' + \frac{1}{\ln t} y = \frac{\cot t}{\ln t}$$

$\text{dom}\left(\frac{1}{\ln t}\right) = (0,1) \cup (1,\infty)$

$\text{dom}\left(\frac{\cot t}{\ln t}\right) = \left[(0,1) \cup (1,\infty)\right] \cap \left[(\pi, \pi)\right] = (0,\pi)$

IC is at $t=2$

So, solution is defined on $(1,\pi)$.

10. Use the integrating factor $\mu(x, y) = xe^x$ to find the general solution of the differential equation.

$$(x + 2) \sin y \, dx + x \cos y \, dy = 0$$

$(x^2e^x + 2xe^x)\sin y\,dx + x^2e^x \cos y\,dy = 0$

$\int M dx = \sin y \int x^2e^x + 2xe^x\,dx = \sin y\,(x^2e^x - 2xe^x + 2e^x + 2xe^x - 2e^x) + h_1(y)$

$= x^2e^x \sin y + h_1(y)$

+	x^2	e^x
−	$2x$	↓
+	2	
	0	

+	$2x$	↓
−	2	
	0	

$\int N dy = x^2e^x \int \cos y\,dy = x^2e^x \sin y + h_2(x)$

So, sol'n is: $\psi(x,y) = \boxed{x^2e^x \sin y = C}$