6.4: DE w/ discontinuous forcing functions

Ex. $2y'' + y' + 2y = g(t)$, $\qquad g(t) = u_5(t) - u_{20}(t) = \begin{cases} 0 & 0 \le t < 5 \\ 1 & 5 \le t < 20 \\ 0 & t \ge 20 \end{cases}$

$y(0) = y'(0) = 0$

$$2\,\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + 2\,\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

$$2\,s^2\,Y(s) - 2s\,y(0) - 2y'(0) + s\,Y(s) - y(0) + 2Y(s) = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$$(2s^2 + s + 2)\,Y(s) = \left(e^{-5s} - e^{-20s}\right)\frac{1}{s} \qquad \text{or}$$

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2+s+2)}$$

Let $H(s) = \dfrac{1}{s(2s^2+s+2)}$, then $Y(s) = e^{-5s}H(s) - e^{-20s}H(s)$

If $h(t) = \mathcal{L}^{-1}\{H(s)\}$, then

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = u_5(t)\,h(t-5) - u_{20}(t)\,h(t-20)$$

Now we just need to find $h(t)$:

$$H(s) = \frac{A}{s} + \frac{Bs + C}{2s^2+s+2}$$

$$\Rightarrow \quad 1 = 2As^2 + As + 2A + Bs^2 + Cs$$

$$\left.\begin{aligned} 1 &= 2A \\ 0 &= A + C \\ 0 &= 2A + B \end{aligned}\right\} \quad A = 1/2,\ B = -1,\ C = -1/2$$

$$\text{So} \quad H(s) = \frac{1}{2}\left(\frac{1}{s}\right) + \underbrace{\frac{-s - 1/2}{2s^2+s+2}}_{\text{complete the square on this.}}$$

$$2\left(s^2 + \tfrac{1}{2}s + 1\right) = 2\left((s + 1/4)^2 + 15/16\right)$$

$$\text{so} \quad H(s) = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\,\frac{(s + 1/4) + 1/4}{(s+1/4)^2 + 15/16}$$

We then get that

$$h(t) = \frac{1}{2} - \frac{1}{2}\left[e^{-t/4}\cos\left(\frac{\sqrt{15}}{4}t\right) + \left(\frac{\sqrt{15}}{15}\right) e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right)\right]$$

The graph of the sol'n is in the book p. 333.

$$\boxed{y(t) = u_5(t)\,h(t-5) - u_{20}(t)\,h(t-20)}$$

One more example:

Ex. (5) $\quad y'' + 3y' + 2y = f(t); \quad y(0) = y'(0) = 0, \quad f(t) = \begin{cases} 1 & 0 \le t < 10 \\ 0 & t \ge 10. \end{cases}$

This becomes:

$$y'' + 3y' + 2y = 1 - u_{10}(t)$$

$$s^2 Y(s) + 3s\,Y(s) + 2Y(s) = \frac{1}{s} - \frac{e^{-10s}}{s}$$

$$(s^2 + 3s + 2)\,Y(s) = \frac{1 - e^{-10s}}{s}$$

$$Y(s) = \frac{1 - e^{-10s}}{s(s^2+3s+2)}$$

$$= H(s) - e^{-10s} H(s), \quad H(s) = \frac{1}{s(s^2+3s+2)}$$

Then $y(t) = h(t) - u_{10}(t)\,h(t-10)$ ... if we know $h(t)$.

$$H(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\rightarrow \quad 1 = A(s+1)(s+2) + B\,s(s+2) + C\,s(s+1)$$

$s = 0: \quad A = 1/2$

$s = -1: \quad B = -1$

$s = -2: \quad C = 1/2$

$$H(s) = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{s+1} + \frac{1}{2}\left(\frac{1}{s+2}\right)$$

Thus $h(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$ and

$$\boxed{y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} + u_{10}(t)\left[\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)}\right]}$$