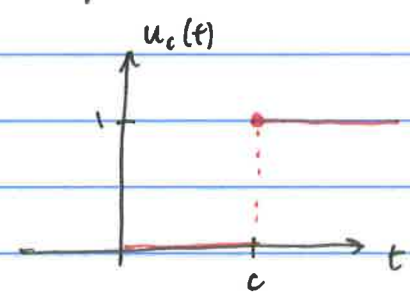
6.3: Step Functions

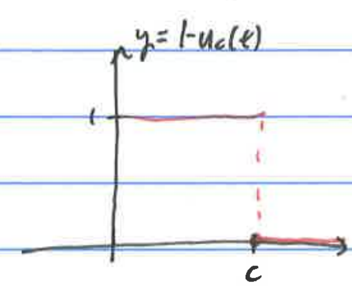
In problems involving the flow of a current or the vibration of a mechanical system we will need to study linear DE with discontinuous or impulse forcing functions.

Def'n. The unit step function or Heaviside function is defined by

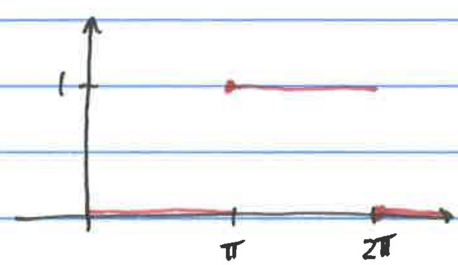
$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}, \quad c \geq 0$$

Graphs: $y = u_c(t)$ $y = 1 - u_c(t)$

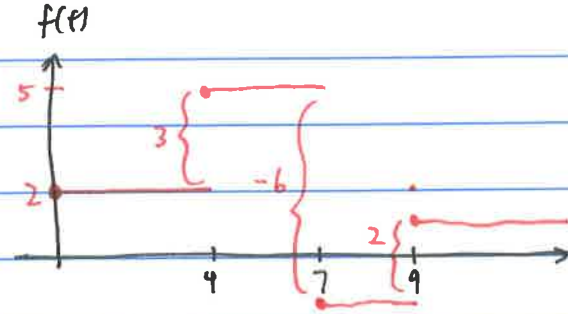




Ex. Let $h(t) = u_\pi(t) - u_{2\pi}(t)$, $t \geq 0$



Ex. $f(t) = \begin{cases} 2 & 0 \le t < 4 \\ 5 & 4 \le t < 7 \\ -1 & 7 \le t < 9 \\ 1 & t \ge 9 \end{cases}$



so $\boxed{f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t)}$

Ex. Find $\mathcal{L}\{u_c(t)\}$ for $c \ge 0$.

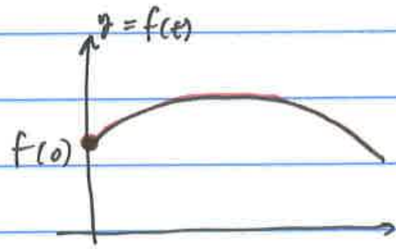
$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\,dt = \int_c^\infty e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_c^\infty$$

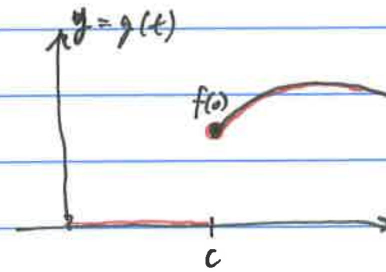
$$= 0 + \frac{1}{s} e^{-cs}, \quad s > 0$$

$$\boxed{\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}}$$

Let f be a function on $t \ge 0$. We want to translate f c units in the positive t direction. (Think: wave)
We form a new function

$$g(t) = \begin{cases} 0 & t < c \\ f(t-c) & t \ge c \end{cases} = u_c(t)\, f(t-c)$$





(*) Thm. If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \ge 0$, and $c > 0$, then

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), \quad s > a.$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

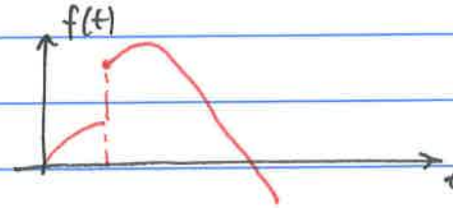
$$u_c(t) f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}.$$

Prove the theorem by making a change of variable $\xi = t-c$ in the integral for $\mathcal{L}\{u_c(t)f(t-c)\}$.

<u>Ex</u>. Let $f(t)=1$ and recall that $\mathcal{L}\{1\} = \frac{1}{s}$.

Compute $\mathcal{L}\{u_c(t)\} = e^{-cs}\mathcal{L}\{1\} = e^{-cs}/s$. $\square$

<u>Ex</u>. $f(t) = \begin{cases} \sin t & 0 \le t < \pi/4 \\ \sin t + \cos(t-\frac{\pi}{4}) & t \ge \pi/4 \end{cases}$



Let $g(t) = u_{\pi/4}(t)\cos(t-\pi/4)$

then $f(t) = \sin t + g(t)$ and

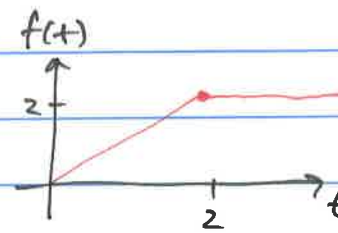
$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{g(t)\}$$
$$= \mathcal{L}\{\sin t\} + e^{-\pi s/4}\mathcal{L}\{\cos t\}$$
$$= \frac{1}{s^2+1} + e^{-s\pi/4}\frac{s}{s^2+1} = \boxed{\frac{1+se^{-s\pi/4}}{s^2+1}}$$

<u>Ex</u>. $F(s) = \frac{1-e^{-2s}}{s^2}$ Find $f(t)$

$$F(s) = \frac{1}{s^2} - e^{-2s}\frac{1}{s^2}$$

$\Rightarrow$

$$f(t) = t - u_2(t)(t-2) = \begin{cases} t & 0 \le t < 2 \\ 2 & t \ge 2 \end{cases}$$



<u>Thm</u>. If $F(s) = \mathcal{L}\{f(t)\}$, $s > a \ge 0$, $c \in \mathbb{R}$, then

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c) \qquad s > a+c.$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}.$$

Ex. $G(s) = \frac{1}{s^2-4s+5}$. Find $g(t)$.

$$G(s) = \frac{1}{(s-2)^2+1} = F(s-2) \quad \text{where} \quad F(u) = \frac{1}{u^2+1}$$

$$\Rightarrow \quad g(t) = e^{2t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \boxed{e^{2t}\sin t}$$