6.2: Solutions of Initial value problems

Thm. Let f be continuous and f' piecewise continuous on any interval $0 \le t \le A$. Suppose there are constants K, a, M such that $|f(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$ and

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$$

Proof.
$$\int_0^A e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^A + s\int_0^A e^{-st} f(t)\,dt$$
$$= e^{-sA} f(A) - e^{0} f(0) + s\int_0^A e^{-st} f(t)\,dt$$

As $A \to \infty$, $|e^{-sA} f(A)| \le K e^{-(s-a)A} \to 0$ for all $s > a$.

Thus
$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0). \quad \square$$

If f, f', and f'' satisfy all of the conditions, then integrating by parts twice yields

$$\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s f(0) - f'(0).$$

In general, we get this Corollary:

Cor. ... Assumptions...
$$\mathcal{L}\{f^{(n)}(t)\} = s^n\,\mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

Thus, the Laplace transform changes differentiation in t to multiplication by s!

We can use the same method to solve a general linear constant coefficient SODE:

$$ay'' + by' + cy = f(t) \qquad (*)$$

we obtain

$$a\left[s^2Y(s) - sy(0) - y'(0)\right] + b\left[sY(s) - y(0)\right] + cY(s) = F(s)$$

or

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2+bs+c} + \frac{F(s)}{as^2+bs+c}$$

Then $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ is the sol'n of $(*)$.

(*) There is a table of Laplace transforms in the book: p. 317.

<u>Ex</u>. $y'' + y = \sin 2t \qquad y(0)=2,\ y'(0)=1$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+2^2}, \quad s>0$$

eventually get: $\boxed{y = \varphi(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t}$

<u>Ex</u>. $y^{(4)} - y = 0 \qquad y(0)=0,\ y'(0)=1,\ y''(0)=y'''(0)=0.$

For PFD choose $s=\pm 1$, $s=0$, then solve for the last one.

equating coefficients at the cubic term.

get: $y = \varphi(t) = \frac{1}{2}\sinh t + \frac{1}{2}\sin t$

6.1, #26!

Ex. **The Gamma Function**

$$\boxed{\Gamma(p+1) = \int_0^\infty e^{-x} x^p \, dx}$$

This integral converges for all p as $x \to \infty$.
For $p<0$, it is also improper as $x \to 0$.
It can be shown that it converges at $x=0$ for $p>-1$.

a) For $p>0$ $\quad \Gamma(p+1) = p\,\Gamma(p)$:

Pf
$$\begin{aligned}
\Gamma(p+1) &= \int_0^\infty e^{-x} x^p \, dx \\
&= \int_0^\infty \frac{d}{dx}\left[-e^{-x}\right] x^p \, dx \\
&= \cancelto{0}{-e^{-x} x^p \Big|_0^\infty} + \int_0^\infty e^{-x} p x^{p-1} \, dx \\
&= p \int_0^\infty e^{-x} x^{p-1} \, dx \\
&= p\,\Gamma(p). \quad \square
\end{aligned}$$

b) $\Gamma(1) = 1$:

Pf
$$\begin{aligned}
\Gamma(1) &= \int_0^\infty e^{-x} x^0 \, dx = \int_0^\infty e^{-x} \, dx \\
&= -e^{-x}\Big|_0^\infty = -e^{-\infty} + e^0 = 0 + 1 = 1 \quad \blacksquare
\end{aligned}$$

c) If p is a positive integer n, then $\Gamma(n+1) = n!$.

Pf By mathematical induction.

$\Gamma(1) = 1$ $\qquad$ by b)

$\Gamma(2) = \Gamma(1+1) = 1 \cdot \Gamma(1) = 1!$

$\Gamma(3) = \Gamma(2+1) = 2 \cdot \Gamma(2) = 2 \cdot 1! = 2!$

Assume this holds for $n \geq 1$

$$\Gamma(n+1) = n!$$

Then $\Gamma(n+2) = \Gamma((n+1)+1) = (n+1)\,\Gamma(n+1)$ by part a)

$= (n+1)\cdot n!$ by ind. hyp.

$= (n+1)!$

Thus, by math. induction, $\Gamma(n+1) = n!$ for all $n \geq 1$. $\square$

Notice $\Gamma(1) = 0! = 1$ by this definition!
Thus, the Γ function is an extension of the factorial function to all numbers $p > -1$.

d) For $p > 0$,

$$\Gamma(p+n) = p(p+1)(p+2)\cdots(p+n-1)\,\Gamma(p)$$

Proof: RE. Integrate by parts n-times. ☺. $\square$

Use this formula and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ to find $\Gamma(\frac{3}{2})$ and $\Gamma(\frac{11}{2})$.

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\cdot \Gamma\left(\tfrac{1}{2}\right) = \boxed{\frac{\sqrt{\pi}}{2}}$$

$$\Gamma\left(\tfrac{11}{2}\right) = \Gamma\left(5+\tfrac{1}{2}\right) = \tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}\cdot\tfrac{7}{2}\cdot\tfrac{9}{2}\,\Gamma\left(\tfrac{1}{2}\right)$$

$$= \boxed{\frac{945}{32}\sqrt{\pi}}$$