Chapter 6: The Laplace Transform

Many engineering and applied problems involve mechanical or electrical systems acted on by discontinuous or impulse forcing terms. The methods of chapter 3 are not well-suited for this type of problem.

The method of Laplace transforms is useful in much more generality, but it works especially well for the types of problems just discussed.

6.1: Definition

Review: Improper integrals:

The Laplace transform will involve an integral like this:

$$\int_a^\infty f(t)\,dt = \lim_{x\to\infty} \int_a^x f(t)\,dt$$

If the integral exists for all $x > a$, ~~then~~ and the limit exists, then the integral $\int_a^\infty f(t)\,dt$ is said to converge. Otherwise, the integral diverges. (fails to exist)

Ex. $f(t) = e^{ct}$, $t \geq 0$, $c > 0$.

$$\int_0^\infty e^{ct}\,dt = \lim_{x\to\infty} \int_0^x e^{ct}\,dt = \lim_{x\to\infty} \frac{1}{c} e^{ct}\Big|_0^x$$
$$= \lim_{x\to\infty} \frac{1}{c}e^{cx} - \frac{1}{c}$$
$$= \infty - \frac{1}{c} = \infty \quad \text{diverges.}$$

If $c<0$, then $\lim_{x\to\infty} \frac{1}{c}e^{cx} = 0$ and the integral equals $-1/c$.

Ex. $f(t) = \frac{1}{t}$, $t \geq 1$

$$\int_1^\infty \frac{1}{t}\,dt = \lim_{x\to\infty} \int_1^x \frac{1}{t}\,dt = \lim_{x\to\infty} \ln t\Big|_1^x$$
$$= \lim_{x\to\infty} \ln x - 0$$
$$= \infty$$

So the integral diverges.

Ex. $f(t) = \frac{1}{t^p}$, $p > 1$, $t \geq 1$

$$\int_1^\infty \frac{1}{t^p}\,dt = \lim_{x\to\infty} \int_1^x t^{-p}\,dt$$

$$= \lim_{x\to\infty} \frac{1}{1-p} t^{1-p}\Big|_1^\infty = \lim_{x\to\infty} \frac{1}{1-p}\left(x^{1-p} - 1\right)$$

$$= \frac{1}{p-1} \quad \text{converges.}$$

If $p < 1$, then the limit DNE $(=\infty)$ and the integral diverges. (p-test).

Recall that if a function is piecewise continuous on an interval, then it is integrable on that integral.
Break up (partition) the interval, then add up all of the integrals.

__Theorem__. If f is piecewise continuous for $t \geq a$, $|f(t)| \leq g(t)$ when $t \geq M > 0$, and $\int_M^\infty g(t)\,dt$ converges, then $\int_a^\infty f(t)\,dt$ also converges.
On the other hand, if $f(t) \geq g(t) \geq 0$ for $t \geq M$ and $\int_M^\infty g(t)\,dt$ diverges, then so does $\int_a^\infty f(t)\,dt$.
(Comparison Thm.)

__The Laplace Transform.__
An __integral__ __transform__ is a relation of the form:

$$F(s) = \int_\alpha^\beta K(s,t)\, f(t)\, dt$$

where $K(s,t)$ is called the __kernel__ of the transform.

There are many such transforms, but we will study the Laplace transform:

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt} = F(s)$$

The kernel of the Laplace transform is $K(s,t) = e^{-st}$.

How will we use the Laplace transform?

1. Use $\mathcal{L}$ to transform an initial value problem for an unknown function f into an algebraic problem for F.
2. Solve the algebraic problem.
3. Invert the transform to recover f.

In general s can (and should) be complex, but we will consider only problems where s is real.

<u>Theorem.</u> Suppose:

1. f is piecewise continuous on the interval $0 \le t \le x$ for any $x > 0$.
2. $|f(t)| \le K e^{at}$ when $t \ge M$. $K, a, M \in \mathbb{R}$ and $K, M > 0$.

Then the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$.

<u>Pf.</u> $\int_0^\infty e^{-st} f(t)\, dt = \int_0^M e^{-st} f(t)\, dt + \int_M^\infty e^{-st} f(t)\, dt$

In the 2^{nd} integral $|e^{-st} f(t)| \le K e^{-st} e^{at} = K e^{(a-s)t}$

Thus $\int_M^\infty K e^{(a-s)t}\, dt$ converges for all $s > a$.

<u>Ex.</u> $f(t) = 1$, $t \ge 0$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt = -\lim_{x\to\infty} \frac{e^{-st}}{s}\Big|_0^x = \frac{1}{s}, \quad s > 0.$$

More examples:

Last time we showed that

1. $$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \frac{1}{s} \qquad s>0$$

2. $$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a} \qquad s>a$$

3. $$\mathcal{L}\{t\} = \int_0^\infty t\,e^{-st}\,dt = \frac{1}{s^2} \qquad s>0$$

Ex(6). Let
$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ k & t = 1 \\ 0 & t > 1 \end{cases} \qquad k \in \mathbb{R}$$

This function represents a unit pulse in engineering, perhaps force or voltage.

f is piecewise continuous, so we can take:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}\,dt = \frac{-e^{-st}}{s}\Big|_0^1 = \frac{1-e^{-s}}{s} \qquad s>0.$$

Notice, this doesn't depend on the value of the function at $t=1$. Even if f is not defined there, $\mathcal{L}\{f\}$ still makes sense.