5.4: Euler Equations; Regular Singular Points

We're still studying equations like

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

But now we want to consider singular pts: that is, pts such that $P(x_0) = 0$.
where P, Q, R have no common factors.

Ex. Euler's Equation. $\quad x^2y'' + \alpha x y' + \beta y = 0, \quad \alpha, \beta \in \mathbb{R}. \qquad (*)$

The only singular point of $(*)$ is $x_0 = 0$.
All other pts are ordinary.
Consider the interval $x > 0$.

Observe that $(x^r)' = r x^{r-1}$ and $(x^r)'' = r(r-1)x^{r-2}$
so if we assume that $(*)$ has solns of the form

$$y = x^r,$$

then the eqn $(*)$ becomes

$$x^2\left[r(r-1)x^{r-2}\right] + \alpha x\left[r x^{r-1}\right] + \beta x^r = 0$$

or

$$x^r\left[r(r-1) + \alpha r + \beta\right] = 0$$

The roots of the eqn $r^2 + (\alpha-1)r + \beta = 0$ are

$$\boxed{r_1, r_2 = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}}$$

Now we can build solutions just like we did in Chapter 3, separating out the cases of real and complex roots.

- <u>Real, distinct roots.</u> Let $F(r) = r^2 + (\alpha-1)r + \beta$.

If $F(r) = 0$ has real roots r_1, r_2, $r_1 \neq r_2$, then $y_1(x) = x^{r_1}$ and $y_2(x) = x^{r_2}$ form a fundamental sol'n set.

Indeed, $W(x^{r_1}, x^{r_2}) = \begin{vmatrix} x^{r_1} & x^{r_2} \\ r_1 x^{r_1-1} & r_2 x^{r_2-1} \end{vmatrix} = r_2 x^{r_1+r_2-1} - r_1 x^{r_1+r_2-1}$

$$= x^{r_1+r_2-1}(r_2 - r_1)$$

Since $r_2 \neq r_1$ and $x > 0$, this Wronskian is nonzero.

(*) Note: If $r \in \mathbb{Q}$ (is rational), then $x^r = e^{r \ln x}$.

<u>Ex.</u> Solve $2x^2 y'' + 3xy' - y = 0$, $x > 0$.

Substituting $y = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$, gives

$$2r(r-1)x^r + 3rx^r - 1x^r = 0$$

$$x^r[2r^2 - 2r + 3r - 1] = 0$$

$$2r^2 + r - 1 = 0$$

$$2r(r+1) - (r+1) = 0$$

$$(2r-1)(r+1) = 0$$

$$r = 1/2, -1$$

So the general sol'n is $y = c_1 x^{1/2} + c_2 x^{-1}$, $x > 0$.

Ex. Equal Roots. If $r = r_1$ has a single, repeating root, then $y_1 = x^{r_1}$ is the only sol'n. The other can be obtained via reduction of order (we actually did some like this), but here we will use a different method.

Since $r_1 = r_2$, then $F(r) = (r - r_1)^2$. Thus, not only does $F(r_1) = 0$, but so does $F'(r) = 2(r - r_1)$, when $r = r_1$.

This suggests differentiating the eq'n:

$$x^r[r(r-1) + \alpha r + \beta]$$
$$= x^r[r^2 + (\alpha - 1)r + \beta] \qquad \text{with respect to } \underline{\underline{r}}.$$

and putting $r = r_1$. Differentiating gives:

$$\frac{\partial}{\partial r} L[x^r] = \frac{\partial}{\partial r}[x^r F(r)]$$

$$L\left[\frac{\partial}{\partial r} x^r\right] = x^r \ln x \underbrace{F(r)}_{=0} + x^r \underbrace{F'(r)}_{=0}$$

$$L[x^r \ln x] = 0$$

Plugging in $r = r_1$ we see that $x^{r_1} \ln x$ is a sol'n.

$$y_2 = x^{r_1} \ln x, \quad x > 0.$$

Taking the Wronskian:

$$W(x^{r_1}, x^{r_1}\ln x) = x^{2r_1 - 1} \neq 0 \quad \text{since } x>0.$$

Thus, the general sol'n is:

$$\boxed{y = (c_1 + c_2 \ln x)\, x^{r_1}}, \quad x>0$$

Ex. Solve: $x^2 y'' + 5xy' + 4y = 0, \quad x>0.$

Plugging in $y = x^r$, $y' = r x^{r-1}$, $y'' = r(r-1)x^{r-2}$, gives

$$r(r-1)x^r + 5r x^r + 4x^r = 0$$

$$x^r\left[r^2 + 4r + 4\right] = 0$$

$$r = -2$$

and the general sol'n is $\boxed{y = (c_1 + c_2 \ln x)x^{-2}}$.

Ex. Complex Roots. Finally, suppose $r_1, r_2 = \lambda \pm i\mu$.

without working through all of the details we get

main identity used is again

$$\boxed{x^r = e^{r\ln x}}$$

$$y = c_1 x^{\lambda + i\mu} + c_2 x^{\lambda - i\mu}$$

$$\boxed{y = c_1 x^{\lambda}\cos(\mu \ln x) + c_2 x^{\lambda}\sin(\mu\ln x)}, \quad x>0.$$

Check that $W\left(x^{\lambda}\cos(\mu\ln x), x^{\lambda}\sin(\mu\ln x)\right) = \mu\, x^{2\lambda - 1}$.

Ex. Solve: $x^2y'' + xy' + y = 0$.

substituting in gives:

$$x^r(r^2+1) = 0$$

$$r = \pm i \quad , \quad \lambda = 0,\ \mu = 1$$

so the soln is $\boxed{y = c_1\cos(\ln x) + c_2\sin(\ln x)}$. $x>0$.

Next: Regular versus Irregular Singular Points.

If x_0 is a singular point, then $P(x_0)=0$ and at least one of $Q(x_0)$ and $R(x_0)$ are not zero.

This means that we can consider how $\frac{Q}{P}$ and $\frac{R}{P}$ behave "at" infinity.

Defn. A point x_0 is called a regular singular point if $P(x_0)=0$ and these two limits exist (and are finite, obviously)

$$\begin{cases} \lim_{x\to x_0} (x-x_0)\dfrac{Q(x)}{P(x)} \\ \lim_{x\to x_0} (x-x_0)^2\dfrac{R(x)}{P(x)} \end{cases}$$

Ex. **Legendre's eqn**

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

Determine whether the singular pts $x_0 = \pm 1$ are regular or not.

$$\lim_{x\to 1} (x-1)\frac{-2x}{1-x^2} = \lim_{x\to 1} \cancel{(x-1)}\frac{+2x}{+(x+1)\cancel{(x-1)}} = \frac{2}{2} = 1$$

~~So $x_0 = 1$ is regular.~~ Not yet.

$$\cancel{\lim_{x\to -1} (x+1)\frac{-2x}{1-x^2}}$$

$$\lim_{x\to 1} (x-1)^2\frac{\alpha(\alpha+1)}{1-x^2} = \lim_{x\to 1} (x-1)\cancel{(x-1)}\frac{\alpha(\alpha+1)}{-\cancel{(x-1)}(x+1)} = \frac{0}{2} = 0$$

Now $x_0 = 1$ is <u>regular</u>.

$x_0 = -1$:

$$\lim_{x\to -1} (x+1)\frac{-2x}{1-x^2} = \lim_{x\to -1} \cancel{(x+1)}\frac{-2x}{\cancel{(x+1)}(1-x)} = \frac{2}{2} = 1$$

$$\lim_{x\to -1} (x+1)^2\frac{\alpha(\alpha+1)}{1-x^2} = \lim_{x\to -1} \frac{(x+1)\,\alpha(\alpha+1)}{1-x} = \frac{0}{2} = 0$$

So $x_0 = -1$ is also <u>regular</u>.

Ex (22) $x^2 y'' + x y' + (x^2 - \nu^2) y = 0$ Bessel's Eqn

$x_0 = 0$ is singular.

$$\lim_{x \to 0} x \frac{x}{x^2} = \lim_{x \to 0} 1 = 1$$

$$\lim_{x \to 0} x^2 \frac{x^2 - \nu^2}{x^2} = \lim_{x \to 0} x^2 - \nu^2 = -\nu^2$$

So $x_0 = 0$ is regular.

Ex (31) $x^2 y'' - 3(\sin x) y' + (1 + x^2) y = 0$

$x_0 = 0$ is singular.

$$\lim_{x \to 0} x \frac{-3(\sin x)}{x^2} = -3 \lim_{x \to 0} \frac{\sin x}{x} = -3$$

$$\lim_{x \to 0} x^2 \frac{1 + x^2}{x^2} = \lim_{x \to 0} 1 + x^2 = 1$$

So $x_0 = 0$ is regular.