5.3. Series Solutions Near an Ordinary Pt, Part II.

We studied series solutions at ordinary points in the last section. We claimed that such series solutions always exists (at least for a small interval) at these points. Now we try to justify this claim.

Suppose that $y = \varphi(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ is a solution of the DE $P(x)y'' + Q(x)y' + R(x)y = 0$ at (near) an ordinary point x_0.

Take m derivatives and set $x = x_0$ to get:

$$\varphi^{(m)}(x_0) = m!\, a_m$$

Thus, to compute each a_n in the series, we must show that $\varphi^{(n)}(x_0)$, $n = 0, 1, 2, \dots$, can be determined from the DE.

Suppose $y = \varphi(x)$ is a sol'n of the DE satisfying the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$, so that $a_0 = y_0$ and $a_1 = y_0'$.

Put φ back in the DE:

$$P(x)\varphi'' + Q(x)\varphi' + R(x)\varphi = 0$$

In the interval around x_0 where φ is defined, we get

$$\varphi''(x) = -p(x)\varphi'(x) - q(x)\varphi(x)$$

Putting $x = x_0$ yields $\varphi''(x_0) = -p(x_0)\varphi'(x_0) - q(x_0)\varphi(x_0)$ (*)

so $\varphi''(x_0) = -p(x_0)\,a_1 - q(x_0)\,a_0$

or

$$2!\,a_2 = -p(x_0)\,a_1 - q(x_0)\,a_0$$

To determine a_3, differentiate $(*)$ to get

$$\varphi'''(x) = -p'(x)\varphi'(x) - p(x)\varphi''(x) - q'(x)\varphi(x) - q(x)\varphi'(x).$$

Putting in $x = x_0$,

$$3!\,a_3 = -\left[p'(x_0)\varphi'(x_0) + p(x_0)\varphi''(x_0) + q'(x_0)\varphi(x_0) + q(x_0)\varphi'(x_0)\right]$$

$$= -\left[p'(x_0)\,a_1 + 2!\,p(x_0)\,a_2 + q'(x_0)\,a_0 + q(x_0)\,a_1\right]$$

$$= -2!\,p(x_0)\,a_2 - \left[p'(x_0) + q(x_0)\right]a_1 - q'(x_0)\,a_0$$

Plugging in $-2!\,a_2 = p(x_0)\,a_1 + q(x_0)a_0$ we get

$$3!\,a_3 = [p(x_0)]^2 a_1 + p(x_0)q(x_0)\,a_0 - \left[p'(x_0) + q(x_0)\right]a_1 - q'(x_0)\,a_0$$

or

$$3!\,a_3 = \left[(p(x_0))^2 - p'(x_0) - q(x_0)\right]a_1 + \left[p(x_0)q(x_0) - q'(x_0)\right]a_0$$

Thus a_3 is determined by the initial data a_0 and a_1.

Similarly, we can continue this process <u>ad infinitum.</u>

The main tool that we use in all of this is the assumption that p and q have infinitely many derivatives.

What we really need to assume is that p and q are analytic at x_0. This will ensure that our solutions converge in a nbd of x_0.

Analytic means that p and q can be represented as power series:

$$p = \sum_{n=0}^{\infty} p_n (x-x_0)^n \quad \text{and} \quad q = \sum_{n=0}^{\infty} q_n (x-x_0)^n$$

Now we can generalize the idea of an ordinary pt:

x_0 is an ordinary point for the DE if p and q are analytic at x_0.

Otherwise, the point is singular.

(Fuchs) Thm. x_0 ~~and~~ an ordinary point of the DE $P(x)y'' + Q(x)y' + R(x)y = 0$, then the general sol'n is $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 y_1(x) + a_1 y_2(x)$, where $a_0, a_1 \in \mathbb{R}$ and y_1 and y_2 are two power series that are analytic at x_0.
The sol'ns y_1 and y_2 form a fund. sol'n set.
(*) Furthermore, the radii of conv. of y_1 and y_2 are at least as large as the radii of conv. of p and q.

To see that these form a fundamental sol'n set notice that

$$y_1(x_0)=1,\quad y_2(x_0)=0,\quad y_1'(x_0)=0,\quad y_2'(x_0)=1$$

so that $W(y_1,y_2)(x_0)=1$.

Ex. What is the radius of convergence of the Taylor series for $\frac{1}{1+x^2}$ about $x=0$.

3 ways: Ratio Test, Geometric Series, zeros of denominator.

Ex. What is the radius of conv. of the ~~pow~~ Taylor series for $\frac{1}{x^2-2x+2}$ about $x=0$? about $x=1$?

$x^2-2x+2=0$ has zeros $x=1\pm i$ which are $\sqrt{2}$ away from the origin in the complex plane.

The distance from 1 to $1\pm i$ in $\mathbb{C}$-plane is 1.

Ex. Determine a lower bound for the radius of convergence of solutions about $x=0$ to:

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0 \qquad \text{Legendre's Eq'n.}$$

$$y'' - \frac{2x}{1-x^2}y' + \frac{\alpha(\alpha+1)}{1-x^2}y = 0$$

$1-x^2=0$ has roots ± 1, which have distance 1 from 0 in $\mathbb{R}$.
thus, the radius of conv. of sol'ns is at least $|x|<1$.

Ex. $(1+x^2)y'' + 2xy' + 4x^2y = 0$
about $x=0$, $x=-1/2$

$1+x^2=0$ has roots $\pm i$ which are 1 unit from 0 in $\mathbb{C}$.
The distance from $x=-1/2$ to $\pm i$ is $\sqrt{1^2+(1/2)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$.
Thus, the sol'n converges for $|x+1/2| < \frac{\sqrt{5}}{2}$. (at least).

An interesting observation:
Suppose $y(0)=y_0$ and $y'(0)=y_0'$ are initial conditions.
Since $1+x^2 \neq 0$, then the FEUT tells us that there exists a unique sol'n to the IVP on $-\infty < x < \infty$. However, Fuchs's Thm ~~tells us that~~ only guarantees a sol'n on $-1<x<1$.
The unique sol'n on $-\infty<x<\infty$ may not have a power series about $x=0$ that converges for all x.

Ex. Can we determine a sol'n about $x=0$ for the DE
$$y'' + (\sin x)y' + (1+x^2)y = 0?$$
If so, what is its radius of conv?
$1+x^2$ is its own p.s. about $x=0$.
$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ has radius of conv. $R=\infty$.

Thus, the DE has power series sol'ns $y = \sum_{n=0}^{\infty} a_n x^n$ for all a_0, a_1 and the series converges for all $x \in \mathbb{R}$!